

1. 2. Propositional Equivalences

Logical Equivalences or Equivalence Rules

| Laws | Formulae |
|----------------------|--|
| Idempotent Laws | $p \wedge p \Leftrightarrow p, p \vee p \Leftrightarrow p$ |
| Associative Laws | $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$ $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ |
| Commutative Laws | $p \wedge q \Leftrightarrow q \wedge p$ $p \vee q \Leftrightarrow q \vee p$ |
| DeMorgan's Laws | $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$ $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$ |
| Distributive Laws | $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ |
| Complement Laws | $p \wedge \neg p \Leftrightarrow F, p \vee \neg p \Leftrightarrow T$ |
| Dominance Laws | $p \vee T \Leftrightarrow T, p \wedge F \Leftrightarrow F$ |
| Identity Laws | $p \wedge T \Leftrightarrow p, p \vee F \Leftrightarrow p$ |
| Absorption Laws | $p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$ |
| Double Negation Laws | $\neg(\neg p) \Leftrightarrow p$ |
| Contra Positive Laws | $p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$ |

| | |
|------------------------------|--|
| Conditional as Disjunction | $p \rightarrow q \Leftrightarrow \neg p \vee q$ |
| Biconditional as Conditional | $p \rightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$ |
| Exportations laws | $p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$ |

1. Determine whether $(\neg Q \wedge (P \rightarrow Q)) \rightarrow \neg P$ is a tautology.

Solution:

| $(\neg Q \wedge (P \rightarrow Q)) \rightarrow \neg P$ | Reason |
|---|---|
| $\Rightarrow (\neg Q \wedge (\neg P \vee Q)) \vee \neg P$ | $P \rightarrow Q \Leftrightarrow \neg P \vee Q$ |
| $\Rightarrow \neg(\neg Q \wedge (\neg P \vee Q)) \vee \neg P$ | $P \rightarrow Q \Leftrightarrow \neg P \vee Q$ |
| $\Rightarrow (Q \vee (P \wedge \neg Q)) \vee \neg P$ | (DeMorgan's law) |
| $\Rightarrow ((Q \vee P) \wedge (Q \vee \neg Q)) \vee \neg P$ | (Distributive law) |
| $\Rightarrow ((Q \vee P) \wedge T) \vee \neg P$ | $P \vee \neg P \Leftrightarrow T$ |
| $\Rightarrow (Q \vee P) \vee \neg P$ | $P \wedge T \Leftrightarrow P$ |
| $\Rightarrow (Q \vee P \vee \neg P)$ | (Associative law) |
| $\Rightarrow (Q \vee T)$ | $P \vee \neg P \Leftrightarrow T$ |
| $\Rightarrow T$ | $P \vee T \Leftrightarrow T$ |

2. Show that the formula $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology.

Solution:

| | |
|---|-----------------------------------|
| $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ | Reason |
| $\Rightarrow Q \vee (P \vee \neg P) \wedge \neg Q$ | (Distributive law) |
| $\Rightarrow (Q \vee (P \vee \neg P)) \vee (Q \vee \neg Q)$ | (Distributive law) |
| $\Rightarrow (Q \vee T) \wedge T$ | $P \vee \neg P \Leftrightarrow T$ |
| $\Rightarrow T \wedge T$ | $P \vee T \Leftrightarrow P$ |

3. Show that the formula $(P \wedge Q) \rightarrow (P \vee Q)$ is a tautology.**Solution:**

| | |
|--|---|
| $(P \wedge Q) \rightarrow (P \vee Q)$ | Reason |
| $\Rightarrow \neg(P \wedge Q) \vee (P \vee Q)$ | $P \rightarrow Q \Leftrightarrow \neg P \vee Q$ |
| $\Rightarrow (\neg P \vee \neg Q) \vee (P \vee Q)$ | (DeMorgan's law) |
| $\Rightarrow (P \vee \neg P) \vee (Q \vee \neg Q)$ | (Associative law) |
| $\Rightarrow T \vee T = T$ | (Negation law) |

4. Show that the formula $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$ **Solution:**

| | |
|--|-----------------------------------|
| $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$ | Reason |
| $\Rightarrow (\neg P \wedge (\neg Q \wedge R)) \vee ((Q \vee P) \wedge R)$ | (Distributive law) |
| $\Rightarrow ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R)$ | (Associative law) |
| $\Rightarrow [(P \vee \neg Q) \vee (Q \vee P)] \wedge R$ | (Distributive law) |
| $\Rightarrow [\neg(P \vee Q) \vee (P \vee Q)] \wedge R$ | (DeMorgan's law) |
| $\Rightarrow T \wedge R$ | $P \vee \neg P \Leftrightarrow T$ |
| $\Rightarrow R$ | $P \wedge T \Leftrightarrow P$ |

Equivalence

Two statement formulas A and B are equivalent iff $A \Leftrightarrow B$ or $A \Leftrightarrow B$ is a tautology.

It is denoted by the symbol $A \Leftrightarrow B$ which is read as “A is equivalent to B.”

Remark:

To prove two statement formulas A and B are equivalent, we can use any one of the following method:

(i) using Truth Table, we show that truth values of both statements formulas A and B are same for each 2^n combinations.

(ii) Assume A. By applying various equivalence rules try to derive B and vice versa.

(iii) Prove $A \Leftrightarrow B$ is a tautology.

1. Show that $\neg(P \vee (\neg P \wedge Q))$ & $\neg P \wedge \neg Q$ are logically equivalent.

Solution:

| $\neg(P \vee (\neg P \wedge Q))$ | Reason |
|---|-------------------------------------|
| $\Leftrightarrow \neg P \wedge (\neg(\neg P \wedge \neg Q))$ | (DeMorgan's law) |
| $\Leftrightarrow \neg P \wedge [\neg(\neg P) \vee \neg Q]$ | (DeMorgan's law) |
| $\Leftrightarrow \neg P \wedge (P \vee \neg Q)$ | (Double Negation law) |
| $\Leftrightarrow (\neg P \wedge P) \vee (\neg P \wedge \neg Q)$ | (Distributive law) |
| $\Leftrightarrow F \vee (\neg P \wedge \neg Q)$ | $\neg P \wedge P \Leftrightarrow F$ |
| $\Leftrightarrow (\neg P \wedge \neg Q) \vee F$ | (Commutative law) |
| $\Leftrightarrow \neg P \wedge \neg Q$ | (identity law) |

Hence $\neg(P \vee (\neg P \wedge Q))$ & $\neg P \wedge \neg Q$ are logically equivalent.

2. Prove that $P \rightarrow Q \Leftrightarrow P \rightarrow (P \wedge Q)$

Solution:

| $P \rightarrow (P \wedge Q)$ | Reason |
|------------------------------|--------|
| | |

| | |
|--|-------------------------------------|
| $\Leftrightarrow \neg P \vee (P \wedge Q)$ | (Conditional as disjunction) |
| $\Leftrightarrow (\neg P \vee P) \wedge (\neg P \wedge Q)$ | (Distributive law) |
| $\Leftrightarrow T \wedge (\neg P \wedge Q)$ | $\neg P \wedge P \Leftrightarrow F$ |
| $\Leftrightarrow \neg P \wedge Q$ | (Identity law) |
| $\Leftrightarrow P \rightarrow Q$ | (Conditional as disjunction) |

3. Prove that $(P \rightarrow R) \wedge (Q \rightarrow R) \Leftrightarrow (P \vee Q) \rightarrow R$

Solution:

| | |
|--|------------------------------|
| $(P \rightarrow R) \wedge (Q \rightarrow R)$ | Reason |
| $\Leftrightarrow (\neg P \wedge R) \wedge (\neg Q \wedge R)$ | (Conditional as disjunction) |
| $\Leftrightarrow (\neg P \wedge \neg Q) \vee R$ | (Distributive law) |
| $\Leftrightarrow \neg(P \vee Q) \vee R$ | (DeMorgan's law) |
| $\Leftrightarrow (P \vee Q) \rightarrow R$ | (Conditional as disjunction) |

4. Prove that $P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$

Solution:

| $P \rightarrow (Q \rightarrow R)$ | Reason |
|---|------------------------------|
| $\Leftrightarrow \neg P \vee (Q \rightarrow R)$ | (Conditional as disjunction) |
| $\Leftrightarrow \neg P \vee (\neg Q \vee R)$ | (Conditional as disjunction) |
| $\Leftrightarrow \neg(\neg P \vee \neg Q) \vee R$ | (Associative law) |
| $\Leftrightarrow \neg(P \wedge Q) \vee R$ | (DeMorgan's law) |
| $\Leftrightarrow (P \wedge Q) \rightarrow R$ | (Conditional as disjunction) |

