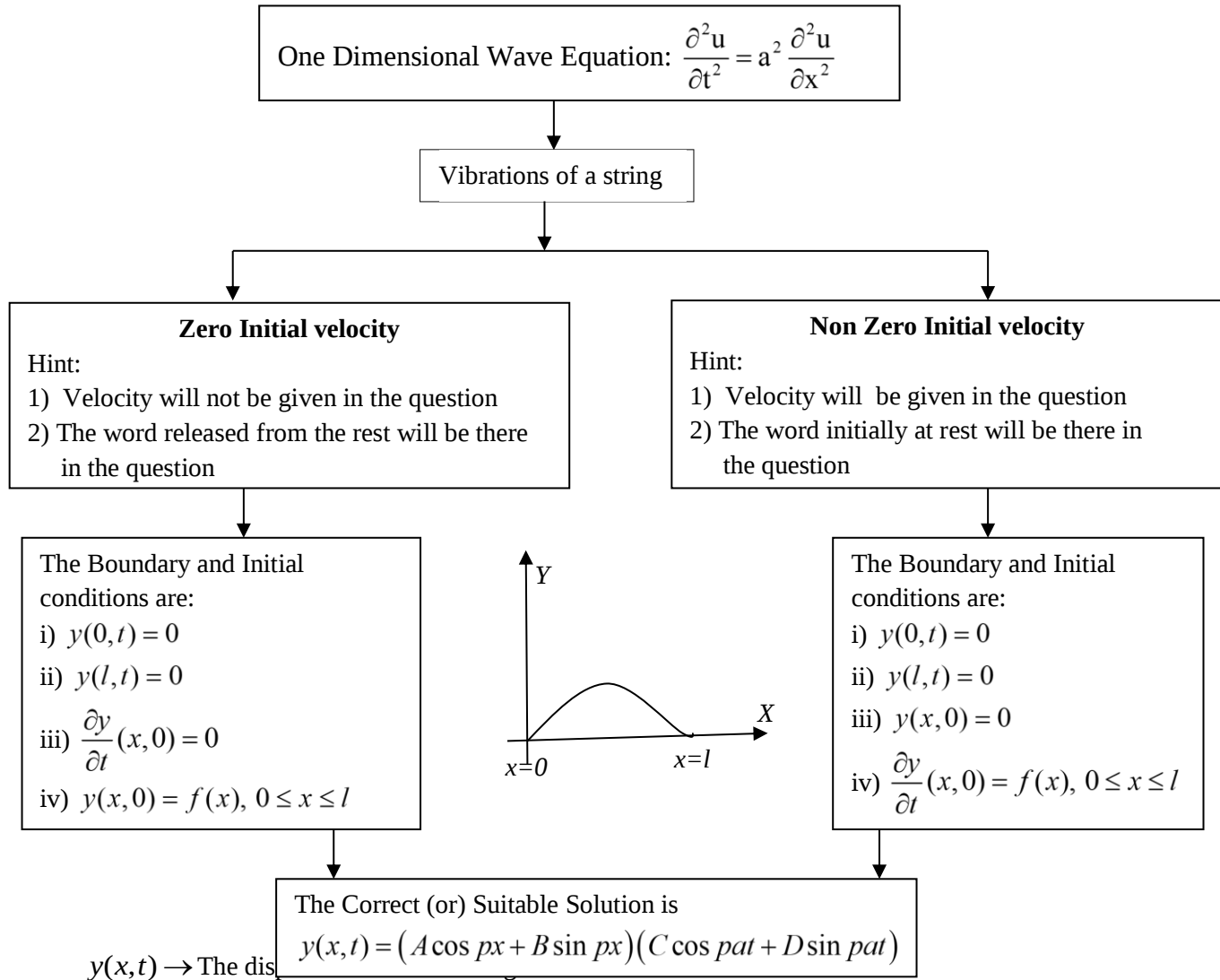


UNIT V APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

5.1 ONE DIMENSIONAL WAVE EQUATION



- 1) In the wave equation $\frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial x^2}$, what does C^2 stands for?

Solution:

One dimensional heat equation is $\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$

$C^2 = T/m$, where T is the tension and m is the mass of the string.

- 2) Write all possible solutions of the transverse vibration of the string in one dimension.

Solution:

$$(i) y(x, t) = (Ae^{px} + Be^{-px}) (Ce^{pat} + De^{-pat})$$

$$(ii) y(x, t) = (A \cos px + B \sin px) (C \cos pat + D \sin pat).$$

$$(iii) y(x, t) = (Ax + B) (Ct + D)$$

PART - B

1. **A uniform string is stretched and fastened to two points 'l' apart. Motion is started by displacing the string into the form of the curve $y = kx(l - x)$ and then releasing it from this position at time $t = 0$. Find the displacement of the point of the string at a distance x from one end at time t .**

Solution:

$$\text{One dimensional wave equation is } \frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \text{ where } a^2 = \frac{T}{m}$$

The correct solution is

$$y(x, t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \text{ ----- (1)}$$

The Boundary and Initial conditions are

$$i) y(0, t) = 0$$

$$ii) y(l, t) = 0$$

$$iii) \frac{\partial y}{\partial t}(x, 0) = 0$$

$$iv) y(x, 0) = f(x) = kx(l - x), \quad 0 < x < l$$

Applying condn (i) in (1)

$$(1) \Rightarrow y(0, t) = (A \cos 0 + \cancel{B \sin 0})(C \cos pat + D \sin pat)$$

$$0 = (A)(C \cos pat + D \sin pat)$$

$$\text{Here } (C \cos pat + D \sin pat) \neq 0 \therefore \boxed{A = 0}$$

Sub $A=0$ in (1)

$$(1) \Rightarrow y(x, t) = (B \sin px)(C \cos pat + D \sin pat) \text{ ----- (2)}$$

Applying condn (ii) in (2)

$$y(l,t) = (B \sin pl)(C \cos pat + D \sin pat)$$

$$0 = (B \sin pl)(C \cos pat + D \sin pat)$$

Here $B \neq 0$, $(C \cos pat + D \sin pat) \neq 0$

$$\therefore \sin pl = 0 \Rightarrow \sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Sub the value of p in (2)

$$(2) \Rightarrow y(x,t) = \left(B \sin \frac{n\pi}{l} x \right) \left(C \cos \frac{n\pi a}{l} t + D \sin \frac{n\pi a}{l} t \right) \text{ ----- (3)}$$

Diff (3) partially w.r.to 't'

$$(2) \Rightarrow \frac{\partial y}{\partial t}(x,t) = \left(B \sin \frac{n\pi}{l} x \right) \left[-C \sin \frac{n\pi a}{l} t \times \left(\frac{n\pi a}{l} \right) + D \cos \frac{n\pi a}{l} t \times \left(\frac{n\pi a}{l} \right) \right]$$

Apply condn. (iii) in the above equation

$$\frac{\partial y}{\partial t}(x,0) = \left(B \sin \frac{n\pi}{l} x \right) \left[\cancel{-C \sin 0 \times \left(\frac{n\pi a}{l} \right)} + D \cos 0 \times \left(\frac{n\pi a}{l} \right) \right]$$

$$0 = \left(B \sin \frac{n\pi}{l} x \right) \left[D \times \left(\frac{n\pi a}{l} \right) \right]$$

Here $B \neq 0$, $\sin \frac{n\pi}{l} x \neq 0$, $\frac{n\pi a}{l} \neq 0$, $\therefore \boxed{D = 0}$

Sub the value of D in (3)

$$(3) \Rightarrow y(x,t) = \left(B \sin \frac{n\pi}{l} x \right) \left(C \cos \frac{n\pi a}{l} t + 0 \right)$$

$$y(x,t) = BC \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t$$

$$y(x,t) = b_1 \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t \quad \text{let } BD = b_1$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t \quad \text{----- (4)}$$

Applying condn (iv) in (4)

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \quad \because \cos 0 = 1$$

Which is half range Fourier sine series in $(0,l)$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \int_0^l kx(l-x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2k}{l} \left[\cancel{(lx - x^2)} \left(\cancel{\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}} \right) - (l-2x) \left(\cancel{\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2 \pi^2}{l^2}}} \right) + (-2) \left(\frac{\cos \frac{n\pi x}{l}}{\frac{n^3 \pi^3}{l^3}} \right) \right]_0^l$$

$$= \frac{2k}{l} \left[\frac{-2l^3}{n^3 \pi^3} \cos \frac{n\pi x}{l} \right]_0^l$$

$$= \frac{-4kl^3}{ln^3 \pi^3} [\cos n\pi - \cos 0]$$

$$= \frac{-4kl^2}{n^3 \pi^3} [(-1)^n - 1]$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8kl^2}{n^3\pi^3} & \text{if } n \text{ is odd} \end{cases}$$

Sub b_n in (4)

$$y(x,t) = \sum_{n=1,3,5,\dots}^{\infty} \frac{8kl^2}{n^3\pi^3} \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t$$

$$(\text{or}) y(x,t) = \frac{8kl^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{(2n-1)\pi}{l} x \cos \frac{(2n-1)\pi a}{l} t$$

2. A string of length $2l$ is fastened at both ends. The midpoint of the string is displaced transversely through a small distance ' b ' and the string is released from the rest in that position. Find an expression for the transverse displacement of the string at any time during the subsequent motion.

Solution:

One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ where $a^2 = \frac{T}{m}$

The correct solution is

$$y(x,t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \text{ ----- (1)}$$

The Boundary and Initial conditions are

Assume $2l=L$

i) $y(0,t) = 0$

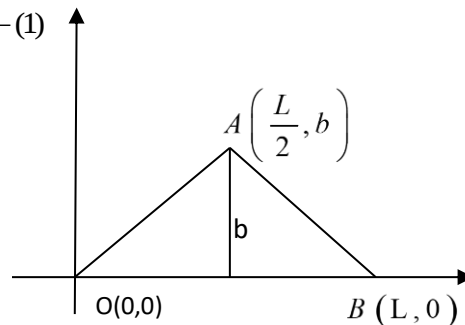
ii) $y(L,t) = 0$

iii) $\frac{\partial y}{\partial t}(x,0) = 0$

iv) $y(x,0) = f(x) = ?$

To find $f(x)$:

The equation of line joining two points is



$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1}$$

Equation of OA is $O(0,0)$ & $A(L/2,b)$:

$$\frac{x-0}{\frac{L}{2}-0} = \frac{y-0}{b-0} \Rightarrow \frac{2x}{L} = \frac{y}{b} \Rightarrow y = \frac{2b}{L}x, \quad 0 < x < \frac{L}{2}$$

Equation of AB is $A(L/2,b)$ & $B(L,0)$:

$$\frac{x-\frac{L}{2}}{L-\frac{L}{2}} = \frac{y-b}{0-b} \Rightarrow \frac{\frac{2x-L}{2}}{\frac{L}{2}} = \frac{y-b}{-b} \Rightarrow \frac{2x-L}{L} = \frac{y-b}{-b}$$

$$-2xb + lb = yL - lb$$

$$-2xb + Lb + Lb = yL \Rightarrow -2xb + 2Lb = yL$$

$$y = \frac{2b}{L}(L-x), \quad \frac{L}{2} < x < L$$

$$y = f(x) = \begin{cases} \frac{2b}{L}x, & 0 < x < \frac{L}{2} \\ \frac{2b}{L}(L-x), & \frac{L}{2} < x < L \end{cases}$$

Applying condn (i) in (1)

$$(1) \Rightarrow y(0,t) = (A \cos 0 + \cancel{B \sin 0})(C \cos pat + D \sin pat)$$

$$0 = (A)(C \cos pat + D \sin pat)$$

Here $(C \cos pat + D \sin pat) \neq 0 \therefore \boxed{A=0}$

Sub $A=0$ in (1)

$$(1) \Rightarrow y(x,t) = (B \sin px)(C \cos pat + D \sin pat) \text{ ----- (2)}$$

Applying condn (ii) in (2)

$$y(L,t) = (B \sin pL)(C \cos pat + D \sin pat)$$

$$0 = (B \sin pL)(C \cos pat + D \sin pat)$$

Here $B \neq 0$, $(C \cos pat + D \sin pat) \neq 0$

$$\therefore \sin pL = 0 \Rightarrow \sin pL = \sin n\pi \Rightarrow pL = n\pi \Rightarrow \boxed{p = \frac{n\pi}{L}}$$

Sub the value of p in (2)

$$(2) \Rightarrow y(x,t) = \left(B \sin \frac{n\pi}{L} x \right) \left(C \cos \frac{n\pi a}{L} t + D \sin \frac{n\pi a}{L} t \right) \text{ ----- (3)}$$

Diff (3) partially w.r.to 't'

$$(2) \Rightarrow \frac{\partial y}{\partial t}(x,t) = \left(B \sin \frac{n\pi}{L} x \right) \left[-C \sin \frac{n\pi a}{L} t \times \left(\frac{n\pi a}{L} \right) + D \cos \frac{n\pi a}{L} t \times \left(\frac{n\pi a}{L} \right) \right]$$

Apply condn. (iii) in the above equation

$$\frac{\partial y}{\partial t}(x,0) = \left(B \sin \frac{n\pi}{L} x \right) \left[-C \sin 0 \times \left(\frac{n\pi a}{L} \right) + D \cos 0 \times \left(\frac{n\pi a}{L} \right) \right]$$

$$0 = \left(B \sin \frac{n\pi}{L} x \right) \left[D \times \left(\frac{n\pi a}{L} \right) \right]$$

Here $B \neq 0$, $\sin \frac{n\pi}{L} x \neq 0$, $\frac{n\pi a}{L} \neq 0$, $\therefore D = 0$

Sub the value of D in (3)

$$(3) \Rightarrow y(x,t) = \left(B \sin \frac{n\pi}{L} x \right) \left(C \cos \frac{n\pi a}{L} t + 0 \right)$$

$$y(x,t) = BC \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t$$

$$y(x,t) = b_1 \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t \quad \text{let } BC = b_1$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t \quad \text{----- (4)}$$

Applying condn (iv) in (4)

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \cos 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \quad \because \cos 0 = 1$$

Which is half range Fourier sine series in $(0,l)$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$= \frac{2}{L} \left[\int_0^{\frac{L}{2}} \frac{2b}{L} x \sin \frac{n\pi x}{L} dx + \int_{\frac{L}{2}}^L \frac{2b}{L} (L-x) \sin \frac{n\pi x}{L} dx \right] \quad \because y = f(x) = \begin{cases} \frac{2b}{L} x, & 0 < x < \frac{L}{2} \\ \frac{2b}{L} (L-x), & \frac{L}{2} < x < L \end{cases}$$

$$= \frac{4b}{L^2} \left[\int_0^{\frac{L}{2}} x \sin \frac{n\pi x}{L} dx + \int_{\frac{L}{2}}^L (L-x) \sin \frac{n\pi x}{L} dx \right]$$

$$= \frac{4b}{L^2} \left\{ \left[(x) \left(\frac{-\cos \frac{n\pi x}{L}}{\frac{n\pi}{L}} \right) - (1) \left(\frac{-\sin \frac{n\pi x}{L}}{\frac{n^2 \pi^2}{L^2}} \right) \right]_0^{\frac{L}{2}} + \left[(L-x) \left(\frac{-\cos \frac{n\pi x}{L}}{\frac{n\pi}{L}} \right) - (-1) \left(\frac{-\sin \frac{n\pi x}{L}}{\frac{n^2 \pi^2}{L^2}} \right) \right]_{\frac{L}{2}}^L \right\}$$

$$= \frac{4b}{L^2} \left\{ \left[-\frac{L}{n\pi} x \cos \frac{n\pi x}{L} + \frac{L^2}{n^2 \pi^2} \sin \frac{n\pi x}{L} \right]_0^{\frac{L}{2}} + \left[-\frac{L}{n\pi} (L-x) \cos \frac{n\pi x}{L} - \frac{L^2}{n^2 \pi^2} \sin \frac{n\pi x}{L} \right]_{\frac{L}{2}}^L \right\}$$

$$= \frac{4b}{L^2} \left\{ \left[\left(-\frac{L}{n\pi} \frac{L}{2} \cos \frac{n\pi}{2} + \frac{L^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right) - (0) \right]_{\frac{L}{2}}^L + \left[(0) - \left(-\frac{L}{n\pi} \left(\frac{L}{2} \right) \cos \frac{n\pi}{2} - \frac{L^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right) \right]_{\frac{L}{2}}^L \right\}$$

$$= \frac{4b}{L^2} \left[\cancel{\frac{L^2}{2n\pi} \cos \frac{n\pi}{2}} + \frac{L^2}{n^2\pi^2} \sin \frac{n\pi}{2} + \cancel{\frac{L^2}{2n\pi} \cos \frac{n\pi}{2}} + \frac{L^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right]$$

$$= \frac{4b}{L^2} \left[\frac{2L^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right]$$

$$b_n = \frac{8b}{n^2\pi^2} \sin \frac{n\pi}{2}$$

Sub b_n in (4)

$$\therefore y(x,t) = \sum_{n=1}^{\infty} \frac{8b}{n^2\pi^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{L} \cos \frac{n\pi at}{L}$$

$$\therefore y(x,t) = \frac{8b}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{2l} \cos \frac{n\pi at}{2l}$$

This is the required displacement.

3. A string of length l is fastened at both ends. The midpoint of the string is displaced transversely through a small distance ' b ' and the string is released from the rest in that position. Find an expression for the transverse displacement of the string at any time during the subsequent motion.

Solution:

Replace L by l in the above problem

4. A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$. If it is released from rest from this position, find the displacement y at any distance x from one end any time t .

Solution:

$$\text{One dimensional wave equation is } \frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \text{ where } a^2 = \frac{T}{m}$$

The correct solution is

$$y(x,t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \text{ ----- (1)}$$

The Boundary and Initial conditions are

i) $y(0,t) = 0$

ii) $y(l,t) = 0$

iii) $\frac{\partial y}{\partial t}(x,0) = 0$

iv) $y(x,0) = f(x) = y_0 \sin^3 \frac{\pi x}{l}, \quad 0 < x < l$

Applying condn (i) in (1)

$$(1) \Rightarrow y(0,t) = (A \cos 0 + \cancel{B \sin 0})(C \cos pat + D \sin pat)$$

$$0 = (A)(C \cos pat + D \sin pat)$$

Here $(C \cos pat + D \sin pat) \neq 0 \therefore \boxed{A=0}$

Sub $A=0$ in (1)

$$(1) \Rightarrow y(x,t) = (B \sin px)(C \cos pat + D \sin pat) \text{ ----- (2)}$$

Applying condn (ii) in (2)

$$y(l,t) = (B \sin pl)(C \cos pat + D \sin pat)$$

$$0 = (B \sin pl)(C \cos pat + D \sin pat)$$

Here $B \neq 0, (C \cos pat + D \sin pat) \neq 0$

$$\therefore \sin pl = 0 \Rightarrow \sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Sub the value of p in (2)

$$(2) \Rightarrow y(x,t) = \left(B \sin \frac{n\pi}{l} x \right) \left(C \cos \frac{n\pi a}{l} t + D \sin \frac{n\pi a}{l} t \right) \text{ ----- (3)}$$

Diff (3) partially w.r.to 't'

$$(2) \Rightarrow \frac{\partial y}{\partial t}(x, t) = \left(B \sin \frac{n\pi}{l} x \right) \left[-C \sin \frac{n\pi a}{l} t \times \left(\frac{n\pi a}{l} \right) + D \cos \frac{n\pi a}{l} t \times \left(\frac{n\pi a}{l} \right) \right]$$

Apply condn. (iii) in the above equation

$$\frac{\partial y}{\partial t}(x, 0) = \left(B \sin \frac{n\pi}{l} x \right) \left[\cancel{-C \sin 0 \times \left(\frac{n\pi a}{l} \right)} + D \cos 0 \times \left(\frac{n\pi a}{l} \right) \right]$$

$$0 = \left(B \sin \frac{n\pi}{l} x \right) \left[D \times \left(\frac{n\pi a}{l} \right) \right]$$

Here $B \neq 0$, $\sin \frac{n\pi}{l} x \neq 0$, $\frac{n\pi a}{l} \neq 0$, $\therefore \boxed{D = 0}$

Sub the value of D in (3)

$$(3) \Rightarrow y(x, t) = \left(B \sin \frac{n\pi}{l} x \right) \left(C \cos \frac{n\pi a}{l} t + 0 \right)$$

$$y(x, t) = BC \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t$$

$$y(x, t) = b_1 \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t \quad \text{let } BD = b_1$$

The most general solution is

$$\boxed{y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t} \text{ ----- (4)}$$

Applying condn (iv) in (4)

$$y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \quad \because \cos 0 = 1$$

$$y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x$$

$$y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \quad \because \sin^3 \theta = \frac{1}{4} [3 \sin \theta - \sin 3\theta]$$

$$y_0 \left\{ \frac{1}{4} \left[3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l} \right] \right\} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x$$

$$\frac{3y_0}{4} \sin \frac{\pi x}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + b_4 \sin \frac{4\pi x}{l} + \dots$$

Equating co-efficients of likely terms on both sides

$$b_1 = \frac{3y_0}{4}; b_2 = 0; b_3 = \frac{y_0}{4}; b_4 = b_5 = b_6 \dots = 0.$$

Sub these values in (4)

$$(4) \Rightarrow y(x, t) = b_1 \sin \frac{\pi x}{l} \cos \frac{\pi at}{l} + b_2 \sin \frac{2\pi x}{l} \cos \frac{2\pi at}{l} + b_3 \sin \frac{3\pi x}{l} \cos \frac{3\pi at}{l} + \dots$$

$$y(x, t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi at}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi at}{l}$$

5. **A tightly stretched string end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $\lambda(lx - x^2)$, then show that the displacement of given string is**

$$y(x, t) = \frac{8\lambda l^3}{a\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin \frac{(2n-1)\pi x}{l} \sin \frac{(2n-1)\pi at}{l}.$$

Solution:

$$\text{One dimensional wave equation is } \frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \text{ where } a^2 = \frac{T}{m}$$

The correct solution is

$$y(x, t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \text{ ----- (1)}$$

The Boundary and Initial conditions are

$$\text{i) } y(0, t) = 0$$

$$\text{ii) } y(l, t) = 0$$

$$\text{iii) } y(x, 0) = 0$$

$$\text{iv) } \frac{\partial y}{\partial t}(x, 0) = \lambda(lx - x^2), \quad 0 < x < l$$

Applying condn (i) in (1)

$$(1) \Rightarrow y(0, t) = (A \cos 0 + \cancel{B \sin 0})(C \cos pat + D \sin pat)$$

$$0 = (A)(C \cos pat + D \sin pat)$$

$$\text{Here } (C \cos pat + D \sin pat) \neq 0 \quad \therefore \boxed{A = 0}$$

Sub $A=0$ in (1)

$$(1) \Rightarrow y(x, t) = (B \sin px)(C \cos pat + D \sin pat) \text{ ----- (2)}$$

Applying condn (ii) in (2)

$$y(l, t) = (B \sin pl)(C \cos pat + D \sin pat)$$

$$0 = (B \sin pl)(C \cos pat + D \sin pat)$$

$$\text{Here } B \neq 0, (C \cos pat + D \sin pat) \neq 0$$

$$\therefore \sin pl = 0 \Rightarrow \sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Sub the value of p in (2)

$$(2) \Rightarrow y(x, t) = \left(B \sin \frac{n\pi}{l} x \right) \left(C \cos \frac{n\pi a}{l} t + D \sin \frac{n\pi a}{l} t \right) \text{ ----- (3)}$$

Apply condn. (iii) in the above equation

$$(3) \Rightarrow y(x, 0) = \left(B \sin \frac{n\pi}{l} x \right) (C \cos 0 + D \sin 0)$$

$$0 = \left(B \sin \frac{n\pi}{l} x \right) (C)$$

$$\text{Here } B \neq 0, \sin \frac{n\pi}{l} x \neq 0, \therefore C = 0$$

Sub the value of C in (3)

$$(3) \Rightarrow y(x, t) = \left(B \sin \frac{n\pi}{l} x \right) \left(D \sin \frac{n\pi a}{l} t \right)$$

$$y(x, t) = BD \sin \frac{n\pi}{l} x \sin \frac{n\pi a}{l} t$$

$$y(x, t) = b_1 \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t \quad \text{let } BD = b_1$$

The most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \sin \frac{n\pi a}{l} t \quad \text{----- (4)}$$

Diff (4) partially w.r.to 't'

$$\frac{\partial y}{\partial t}(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t \times \left(\frac{n\pi a}{l} \right)$$

Applying condn (iv) in the above eqn.

$$\frac{\partial y}{\partial t}(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos 0 \times \left(\frac{n\pi a}{l} \right)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \times \left(\frac{n\pi a}{l} \right) \quad \because \cos 0 = 1$$

$$y(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \quad \text{let } B_n = b_n \frac{n\pi a}{l}$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{l} x$$

Which is half range Fourier sine series in $(0, l)$

$$B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \int_0^l \lambda x(l-x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2\lambda}{l} \left[\cancel{(lx - x^2)} \left(\cancel{\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}} \right) - (l-2x) \left(\cancel{\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2 \pi^2}{l^2}}} \right) + (-2) \left(\frac{\cos \frac{n\pi x}{l}}{\frac{n^3 \pi^3}{l^3}} \right) \right]_0^l$$

$$= \frac{2\lambda}{l} \left[\frac{-2l^3}{n^3 \pi^3} \cos \frac{n\pi x}{l} \right]_0^l$$

$$= \frac{-4\lambda l^3}{ln^3 \pi^3} [\cos n\pi - \cos 0]$$

$$b_n \frac{n\pi a}{l} = \frac{-4\lambda l^2}{n^3 \pi^3} [(-1)^n - 1]$$

$$b_n = \frac{-4\lambda l^3}{n^4 \pi^4 a} [(-1)^n - 1]$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8\lambda l^3}{n^4 \pi^4 a} & \text{if } n \text{ is odd} \end{cases}$$

Sub b_n in (4)

$$y(x, t) = \sum_{n=1,3,5,\dots}^{\infty} \frac{8\lambda l^3}{n^4 \pi^4 a} \sin \frac{n\pi}{l} x \sin \frac{n\pi a}{l} t$$

$$(or) y(x,t) = \frac{8\lambda l^3}{\pi^4 a} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin \frac{(2n-1)\pi}{l} x \sin \frac{(2n-1)\pi a}{l} t$$

6. If a string of length l is initially at rest in its equilibrium position whose ends are fixed and each of its points

is given a velocity v such that $v = \begin{cases} cx; 0 < x < \frac{l}{2} \\ c(l-x); \frac{l}{2} < x < l \end{cases}$, find the displacement of the string at any time t .

Solution:

One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ where $a^2 = \frac{T}{m}$

The correct solution is

$$y(x,t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \text{ ---- (1)}$$

The Boundary and Initial conditions are

i) $y(0,t) = 0$

ii) $y(l,t) = 0$

iii) $y(x,0) = 0$

iv) $\frac{\partial y}{\partial t}(x,0) = \begin{cases} cx; 0 < x < \frac{l}{2} \\ c(l-x); \frac{l}{2} < x < l \end{cases}$

Applying condn (i) in (1)

$$(1) \Rightarrow y(0,t) = (A \cos 0 + \cancel{B \sin 0})(C \cos pat + D \sin pat)$$

$$0 = (A)(C \cos pat + D \sin pat)$$

Here $(C \cos pat + D \sin pat) \neq 0 \quad \therefore A = 0$

Sub $A=0$ in (1)

$$(1) \Rightarrow y(x,t) = (B \sin px)(C \cos pat + D \sin pat) \text{ ----- (2)}$$

Applying condn (ii) in (2)

$$y(l,t) = (B \sin pl)(C \cos pat + D \sin pat)$$

$$0 = (B \sin pl)(C \cos pat + D \sin pat)$$

Here $B \neq 0$, $(C \cos pat + D \sin pat) \neq 0$

$$\therefore \sin pl = 0 \Rightarrow \sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Sub the value of p in (2)

$$(2) \Rightarrow y(x,t) = \left(B \sin \frac{n\pi}{l} x \right) \left(C \cos \frac{n\pi a}{l} t + D \sin \frac{n\pi a}{l} t \right) \text{ ----- (3)}$$

Apply condn. (iii) in the above equation

$$(3) \Rightarrow y(x,0) = \left(B \sin \frac{n\pi}{l} x \right) (C \cos 0 + \cancel{D \sin 0})$$

$$0 = \left(B \sin \frac{n\pi}{l} x \right) (C)$$

Here $B \neq 0$, $\sin \frac{n\pi}{l} x \neq 0$, $\therefore C = 0$

Sub the value of C in (3)

$$(3) \Rightarrow y(x,t) = \left(B \sin \frac{n\pi}{l} x \right) \left(D \sin \frac{n\pi a}{l} t \right)$$

$$y(x,t) = BD \sin \frac{n\pi}{l} x \sin \frac{n\pi a}{l} t$$

$$y(x,t) = b_1 \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t \quad \text{let } BD = b_1$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \sin \frac{n\pi a}{l} t \text{ ----- (4)}$$

Diff (4) partially w.r.to 't'

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t \times \left(\frac{n\pi a}{l} \right)$$

Applying condn (iv) in the above eqn.

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos 0 \times \left(\frac{n\pi a}{l} \right)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \times \left(\frac{n\pi a}{l} \right) \quad \because \cos 0 = 1$$

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \quad \text{let } B_n = b_n \frac{n\pi a}{l}$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{l} x$$

Which is half range Fourier sine series in (0,l)

$$B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \left[\int_0^{\frac{l}{2}} cx \sin \frac{n\pi x}{l} dx + \int_{\frac{l}{2}}^l c(l-x) \sin \frac{n\pi x}{l} dx \right] \quad \because f(x) = \begin{cases} cx; & 0 < x < \frac{l}{2} \\ c(l-x); & \frac{l}{2} < x < l \end{cases}$$

$$= \frac{2c}{l} \left[\int_0^{\frac{l}{2}} x \sin \frac{n\pi x}{l} dx + \int_{\frac{l}{2}}^l (l-x) \sin \frac{n\pi x}{l} dx \right]$$

$$\begin{aligned}
&= \frac{2c}{l} \left\{ \left[(x) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (1) \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2 \pi^2}{l^2}} \right) \right]_0^{\frac{l}{2}} + \left[(l-x) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (-1) \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2 \pi^2}{l^2}} \right) \right]_{\frac{l}{2}}^l \right\} \\
&= \frac{2c}{l} \left\{ \left[-\frac{l}{n\pi} x \cos \frac{n\pi x}{l} + \frac{l^2}{n^2 \pi^2} \sin \frac{n\pi x}{l} \right]_0^{\frac{l}{2}} + \left[-\frac{l}{n\pi} (l-x) \cos \frac{n\pi x}{l} - \frac{l^2}{n^2 \pi^2} \sin \frac{n\pi x}{l} \right]_{\frac{l}{2}}^l \right\} \\
&= \frac{2c}{l} \left\{ \left[\left(-\frac{l}{n\pi} \frac{l}{2} \cos \frac{n\pi}{2} + \frac{l^2}{n^2 \pi^2} \sin \frac{n\pi}{2} \right) - (0) \right]_0^{\frac{l}{2}} + \left[(0) - \left(-\frac{l}{n\pi} \left(\frac{l}{2} \right) \cos \frac{n\pi}{2} - \frac{l^2}{n^2 \pi^2} \sin \frac{n\pi}{2} \right) \right]_{\frac{l}{2}}^l \right\} \\
&= \frac{2c}{l} \left[\cancel{-\frac{l^2}{2n\pi} \cos \frac{n\pi}{2}} + \frac{l^2}{n^2 \pi^2} \sin \frac{n\pi}{2} + \cancel{\frac{l^2}{2n\pi} \cos \frac{n\pi}{2}} + \frac{l^2}{n^2 \pi^2} \sin \frac{n\pi}{2} \right] \\
&= \frac{2c}{l} \left[\frac{2l^2}{n^2 \pi^2} \sin \frac{n\pi}{2} \right]
\end{aligned}$$

$$b_n \frac{n\pi a}{l} = \frac{4cl}{n^2 \pi^2} \sin \frac{n\pi}{2} \quad \therefore B_n = b_n \frac{n\pi a}{l}$$

$$b_n = \frac{4cl^2}{n^3 \pi^3 a} \sin \frac{n\pi}{2}$$

Sub b_n in (4)

$$\therefore y(x, t) = \sum_{n=1}^{\infty} \frac{4cl^2}{n^3 \pi^3 a} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$$

$$\therefore y(x, t) = \frac{4cl^2}{\pi^3 a} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$$

7. If a string of length of l is initially at rest in its equilibrium position and each of its point is given the velocity $\frac{\partial y}{\partial t}(x, 0) = V_0 \sin^3 \frac{\pi x}{l}$; $0 < x < l$. Determine the displacement function $y(x, t)$.

Solution:

One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ where $a^2 = \frac{T}{m}$

The correct solution is

$$y(x,t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \text{ ----- (1)}$$

The Boundary and Initial conditions are

i) $y(0,t) = 0$

ii) $y(l,t) = 0$

iii) $y(x,0) = 0$

iv) $\frac{\partial y}{\partial t}(x,0) = V_0 \sin^3 \frac{\pi x}{l}, 0 < x < l$

Applying condn (i) in (1)

$$(1) \Rightarrow y(0,t) = (A \cos 0 + \cancel{B \sin 0})(C \cos pat + D \sin pat)$$

$$0 = (A)(C \cos pat + D \sin pat)$$

Here $(C \cos pat + D \sin pat) \neq 0 \therefore \boxed{A = 0}$

Sub $A=0$ in (1)

$$(1) \Rightarrow y(x,t) = (B \sin px)(C \cos pat + D \sin pat) \text{ ----- (2)}$$

Applying condn (ii) in (2)

$$y(l,t) = (B \sin pl)(C \cos pat + D \sin pat)$$

$$0 = (B \sin pl)(C \cos pat + D \sin pat)$$

Here $B \neq 0, (C \cos pat + D \sin pat) \neq 0$

$$\therefore \sin pl = 0 \Rightarrow \sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Sub the value of p in (2)

$$(2) \Rightarrow y(x, t) = \left(B \sin \frac{n\pi}{l} x \right) \left(C \cos \frac{n\pi a}{l} t + D \sin \frac{n\pi a}{l} t \right) \text{ ----- (3)}$$

Apply condn. (iii) in the above equation

$$(3) \Rightarrow y(x, 0) = \left(B \sin \frac{n\pi}{l} x \right) (C \cos 0 + \cancel{D \sin 0})$$

$$0 = \left(B \sin \frac{n\pi}{l} x \right) (C)$$

Here $B \neq 0$, $\sin \frac{n\pi}{l} x \neq 0$, $\therefore \boxed{C = 0}$

Sub the value of C in (3)

$$(3) \Rightarrow y(x, t) = \left(B \sin \frac{n\pi}{l} x \right) \left(D \sin \frac{n\pi a}{l} t \right)$$

$$y(x, t) = BD \sin \frac{n\pi}{l} x \sin \frac{n\pi a}{l} t$$

$$y(x, t) = b_1 \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t \quad \text{let } BD = b_1$$

The most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \sin \frac{n\pi a}{l} t \text{ ----- (4)}$$

Diff (4) partially w.r.to 't'

$$\frac{\partial y}{\partial t}(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos \frac{n\pi a}{l} t \times \left(\frac{n\pi a}{l} \right)$$

Applying condn (iv) in the above eqn.

$$\frac{\partial y}{\partial t}(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos 0 \times \left(\frac{n\pi a}{l} \right)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \times \left(\frac{n\pi a}{l} \right) \quad \because \cos 0 = 1$$

$$V_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \frac{n\pi a}{l} \sin \frac{n\pi x}{l}$$

$$V_0 \left\{ \frac{1}{4} \left[3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l} \right] \right\} = \sum_{n=1}^{\infty} b_n \frac{n\pi a}{l} \sin \frac{n\pi x}{l} \quad \because \sin^3 \theta = \frac{1}{4} [3 \sin \theta - \sin 3\theta]$$

$$\frac{3V_0}{4} \sin \frac{\pi x}{l} - \frac{V_0}{4} \sin \frac{3\pi x}{l} = b_1 \frac{\pi a}{l} \sin \frac{\pi x}{l} + b_2 \frac{2\pi a}{l} \sin \frac{2\pi x}{l} + b_3 \frac{3\pi a}{l} \sin \frac{3\pi x}{l} + b_4 \frac{4\pi a}{l} \sin \frac{4\pi x}{l} + \dots$$

Equating co-efficients of likely terms on both sides

$$b_1 \frac{\pi a}{l} = \frac{3V_0}{4}; b_2 \frac{2\pi a}{l} = 0; b_3 \frac{3\pi a}{l} = \frac{-V_0}{4}; b_4 = b_5 = b_6 \dots = 0.$$

$$\boxed{b_1 = \frac{3V_0 l}{4\pi a}; b_2 = 0; b_3 = \frac{-V_0 l}{12\pi a}; b_4 = b_5 = b_6 \dots = 0.}$$

Sub these values in (4)

$$(4) \Rightarrow y(x, t) = b_1 \sin \frac{\pi x}{l} \cos \frac{\pi a t}{l} + \cancel{b_2 \sin \frac{2\pi x}{l} \cos \frac{2\pi a t}{l}} + b_3 \sin \frac{3\pi x}{l} \cos \frac{3\pi a t}{l} + \dots$$

$$\boxed{y(x, t) = \frac{3V_0 l}{4\pi a} \sin \frac{\pi x}{l} \cos \frac{\pi a t}{l} - \frac{V_0 l}{12\pi a} \sin \frac{3\pi x}{l} \cos \frac{3\pi a t}{l}}$$