



ROHINI COLLEGE OF ENGINEERING & TECHNOLOGY

DEPARTMENT OF MATHEMATICS



UNIT - IV : FOURIER SERIES

4.1 GENERAL FORM OF FOURIER SERIES

PART A

1. State Dirichlet's conditions for the existence of Fourier series of $f(x)$ in the interval $(0, 2\pi)$.

A function $f(x)$ can be expanded as a Fourier series in the interval $(0, 2\pi)$ if the following conditions are satisfied.

- (i) $f(x)$ is periodic, single valued and finite in $(0, 2\pi)$
- (ii) $f(x)$ has only finite number of finite discontinuities and no infinite discontinuities in $(0, 2\pi)$.
- (iii) $f(x)$ has only finite number of maxima and minima in $(0, 2\pi)$

2. Does $f(x) = \tan x$ possess a Fourier expansion?

Solution:

$\tan x$ does not possess a Fourier expansion because the function $f(x) = \tan x = \frac{\sin x}{\cos x}$ has the infinite discontinuity at the point $x = \frac{\pi}{2}$.

3. Determine the value of a_n & a_0 in the Fourier series expansion of $f(x) = x^3$ in $-\pi < x < \pi$

Solution:

$$f(x) = x^3 \Rightarrow f(-x) = (-x)^3 = -x^3 = -f(x) \Rightarrow f(x) \text{ is an odd function } \therefore a_n = a_0 = 0$$

4. Find the Fourier constant b_n for $x \sin x$ in $-\pi < x < \pi$, when expressed as a Fourier series.

Solution:

$$f(x) = x \sin x, -\pi < x < \pi$$

$$f(-x) = (-x) \sin(-x) = x \sin x = f(x)$$

$\therefore f(x)$ is an even function $\therefore b_n = 0$

5. Find the constant term of the Fourier series for the function $f(x) = x^2$, $-\pi < x < \pi$

Solution:

$$f(x) = x^2, -\pi < x < \pi$$

$$f(-x) = (-x)^2 = x^2 = f(x)$$

$\therefore f(x)$ is an even function

$$l = \frac{U.L - L.L}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} = \pi$$

$$a_0 = \frac{2}{l} \int_0^l f(x) dx = \frac{2}{\pi} \int_0^\pi x^2 dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^\pi = \frac{2}{\pi} \left[\frac{\pi^3}{3} - 0 \right] = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}$$

$$\therefore \text{Constant term} = \frac{a_0}{2} = \frac{\pi^2}{3}$$

6. Find the root mean square value of the function $f(x) = x$ in $(0, l)$

$$\text{RMS value} = \bar{y} = \sqrt{\frac{1}{b-a} \int_a^b [f(x)]^2 dx} = \sqrt{\frac{1}{l-0} \int_0^l x^2 dx} = \sqrt{\frac{1}{l} \left[\frac{x^3}{3} \right]_0^l} = \sqrt{\frac{l^3}{3l}} = \frac{l}{\sqrt{3}}$$

7. What do you mean by Harmonic analysis?

The process of finding the harmonics in the Fourier series expansion of a function numerically is known as harmonic analysis.

8. Find the constant term in the expression of $\cos^2 x$ as a Fourier series in the interval $(-\pi, \pi)$.

Solution:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 x dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{1 + \cos 2x}{2} \right) dx = \frac{1}{2\pi} \left[x + \frac{\sin 2x}{2} \right]_{-\pi}^{\pi} a_0 = \frac{1}{2\pi} [(\pi) - (-\pi)] = \frac{2\pi}{2\pi} = 1$$

$$\therefore \text{Constant term} = \boxed{\frac{a_0}{2} = \frac{1}{2}}$$

PART B

1.

Find the Fourier series $f(x) = \left(\frac{\pi - x}{2} \right)^2$ in $0 < x < 2\pi$. Hence show that

$$(i) \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6} \quad (ii) \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}.$$

Solution:

$$\text{Given } f(x) = \left(\frac{\pi - x}{2} \right)^2 = \frac{1}{4} (\pi - x)^2 = \frac{1}{4} (\pi^2 - 2\pi x + x^2), 0 < x < 2\pi$$

$$\text{General Fourier is } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{2\pi - 0}{2} = \frac{2\pi}{2} = \pi \quad \therefore \ell = \pi$$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx} \text{----- (1)}$$

To Find a_0 :

$$\begin{aligned} a_0 &= \frac{1}{\ell} \int_0^{2\ell} f(x) dx = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{4} (\pi^2 - 2\pi x + x^2) dx = \frac{1}{4\pi} \int_0^{2\pi} (\pi^2 - 2\pi x + x^2) dx \\ &= \frac{1}{4\pi} \left[\pi^2 x - \frac{2\pi x^2}{2} + \frac{x^3}{3} \right]_0^{2\pi} = \frac{1}{4\pi} \left[\left(2\pi^3 - 4\pi^3 + \frac{8\pi^3}{3} \right) - (0) \right] \\ &= \frac{\pi^3}{4\pi} \left[-2 + \frac{8}{3} \right] = \frac{\pi^2}{4} \left[\frac{-6 + 8}{3} \right] = \frac{\pi^2}{4} \left[\frac{2}{3} \right] \end{aligned}$$

$$\boxed{a_0 = \frac{\pi^2}{6}}$$

To Find a_n

$$\begin{aligned} a_n &= \frac{1}{\ell} \int_0^{2\ell} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{4} (\pi^2 - 2\pi x + x^2) \cos nx dx = \frac{1}{4\pi} \int_0^{2\pi} (\pi^2 - 2\pi x + x^2) \cos nx dx \\ &= \frac{1}{4\pi} \left[\left(\pi^2 - 2\pi x + x^2 \right) \left(\frac{\sin nx}{n} \right) - (-2\pi + 2x) \left(\frac{-\cos nx}{n^2} \right) + (2) \left(\frac{-\sin nx}{n^3} \right) \right]_0^{2\pi} \\ &= \frac{1}{4\pi} \left[\frac{1}{n^2} (-2\pi + 2x) \cos nx \right]_0^{2\pi} \\ &= \frac{1}{4\pi n^2} [(2\pi \cos 2n\pi) - (-2\pi \cos 0)] \\ a_n &= \frac{1}{4\pi n^2} [2\pi + 2\pi] = \frac{4\pi}{4\pi n^2} \quad \because \cos 2n\pi = 1 \text{ \& } \cos 0 = 1 \end{aligned}$$

$$a_n = \frac{1}{n^2}$$

To find b_n :

$$\begin{aligned} b_n &= \frac{1}{l} \int_0^{2l} f(x) \sin \frac{n\pi x}{l} dx \\ &= \frac{1}{\ell} \int_0^{2\ell} f(x) \sin nx dx = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{4} (\pi^2 - 2\pi x + x^2) \sin nx dx = \frac{1}{4\pi} \int_0^{2\pi} (\pi^2 - 2\pi x + x^2) \sin nx dx \\ &= \frac{1}{4\pi} \left[(\pi^2 - 2\pi x + x^2) \left(\frac{-\cos nx}{n} \right) - (-2\pi + 2x) \left(\frac{-\sin nx}{n^2} \right) + (2) \left(\frac{\cos nx}{n^3} \right) \right]_0^{2\pi} \\ &= \frac{1}{4\pi} \left[\frac{-1}{n} (\pi^2 - 2\pi x + x^2) \cos nx + \frac{2}{n^3} \cos nx \right]_0^{2\pi} \\ &= \frac{1}{4\pi} \left[\left(\frac{-1}{n} (\pi^2 - 4\pi^2 + 4\pi^2) \cos 2n\pi + \frac{2}{n^3} \cos 2n\pi \right) - \left(\frac{-1}{n} \pi^2 \cos 0 + \frac{2}{n^3} \cos 0 \right) \right] \\ &= \frac{1}{4\pi} \left(\frac{-\pi^2}{n} + \frac{-\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3} \right) \end{aligned}$$

$$b_n = 0$$

Substitute a_0, a_n, b_n in (1)

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx + \sum_{n=1}^{\infty} 0 \sin nx$$

$$\therefore f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx \quad \text{-----(2)}$$

Deduction:

(i) Let $x=0$ be a point of discontinuity

$$(2) \Rightarrow f(0) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos 0 \quad \because f(0) = \frac{f(0) + f(2\pi)}{2} = \frac{\frac{\pi^2}{4} + \frac{\pi^2}{4}}{2} = \frac{2 \frac{\pi^2}{4}}{2} = \frac{\pi^2}{4}$$

$$\therefore f(x) = \left(\frac{\pi - x}{2} \right)^2$$

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{4} - \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{4\pi^2}{48} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \Rightarrow \boxed{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}}$$

ii) Let $x=\pi$ be a point of continuity

$$f(\pi) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi \quad \because f(x) = \left(\frac{\pi - x}{2} \right)^2 \Rightarrow f(0) = \left(\frac{\pi - \pi}{2} \right)^2 = 0$$

$$0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{12} = \frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots$$

$$\boxed{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}}$$

2.

Obtain the Fourier series for $f(x)$ of period $2l$ and defined as follows $f(x) = \begin{cases} l-x & , 0 < x \leq l \\ 0 & , l \leq x < 2l \end{cases}$. Hence

deduce that (i) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$ (ii) $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$

Solution: General Fourier is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{l - (-l)}{2} = \frac{2l}{2} = l \quad \therefore \ell = l$$

To find a_0

$$a_0 = \frac{1}{\ell} \int_0^{2\ell} f(x) dx = \frac{1}{\ell} \left[\int_0^{\ell} (\ell - x) dx + \int_{\ell}^{2\ell} 0 dx \right]$$

$$= \frac{1}{\ell} \left[lx - \frac{x^2}{2} \right]_0^{\ell} = \frac{1}{\ell} \left[l^2 - \frac{l^2}{2} \right] = \frac{1}{\ell} \left[\frac{l^2}{2} \right]$$

$$\boxed{a_0 = \frac{\ell}{2}}$$

$$a_n = \frac{1}{\ell} \int_0^{2\ell} f(x) \cos \frac{n\pi x}{\ell} dx = \frac{1}{\ell} \left[\int_0^{\ell} (\ell - x) \cos \frac{n\pi x}{\ell} dx + \int_{\ell}^{2\ell} 0 dx \right]$$

$$= \frac{1}{\ell} \left[(\ell - x) \left(\frac{\sin \frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}} \right) - (-1) \left(\frac{-\cos \frac{n\pi x}{\ell}}{\frac{n^2 \pi^2}{\ell^2}} \right) \right]_0^{\ell}$$

$$= \frac{1}{\ell} \left[\frac{-l^2}{n^2 \pi^2} \cos \frac{n\pi x}{\ell} \right]_0^l$$

$$= \frac{-\ell^2}{l n^2 \pi^2} [\cos n\pi - \cos 0] = \frac{\ell}{n^2 \pi^2} [1 - (-1)^n]$$

$$\boxed{a_n = \begin{cases} \frac{2\ell}{n^2 \pi^2}, & n=1, 3, 5, \dots \\ 0 & , n=2, 4, 6, \dots \end{cases}}$$

$$b_n = \frac{1}{\ell} \int_0^{2\ell} f(x) \sin \frac{n\pi x}{\ell} dx = \frac{1}{\ell} \left[\int_0^{\ell} (\ell - x) \sin \frac{n\pi x}{\ell} dx + \int_{\ell}^{2\ell} 0 dx \right]$$

$$b_n = \frac{1}{\ell} \left[(\ell - x) \left(\frac{-\cos \frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}} \right) - (-1) \left(\frac{-\sin \frac{n\pi x}{\ell}}{\frac{n^2 \pi^2}{\ell^2}} \right) \right]_0^\ell$$

$$= \frac{1}{\ell} \left[\frac{-l}{n\pi} (l - x) \cos \frac{n\pi x}{\ell} \right]_0^l = \frac{-l}{\ell n\pi} [0 - l \cos 0] = \frac{l^2}{\ell n\pi}$$

$$\boxed{b_n = \frac{\ell}{n\pi}}$$

Substitute a_0, a_n, b_n in (1)

$$f(x) = \frac{\ell}{4} + \sum_{n=1,3,5}^{\infty} \frac{2\ell}{n^2 \pi^2} \cos \frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} \frac{\ell}{n\pi} \sin \frac{n\pi x}{\ell}$$

$$\therefore f(x) = \frac{\ell}{4} + \frac{2\ell}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} \cos \frac{n\pi x}{\ell} + \frac{\ell}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{\ell} \quad \text{----- (2)}$$

Deduction:

i) Let $x = \ell/2$ be a point of continuity

$$(2) \Rightarrow f(\ell/2) = \frac{\ell}{4} + \frac{2\ell}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} \cos \frac{n\pi}{2} + \frac{\ell}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \quad \because f(x) = \begin{cases} l-x, & 0 < x \leq l \\ 0, & l \leq x < 2l \end{cases}$$

$$\Rightarrow \frac{\ell}{2} = \frac{\ell}{4} + \frac{2\ell}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} \cos \frac{n\pi}{2} + \frac{\ell}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \quad \because f\left(\frac{l}{2}\right) = l - \frac{l}{2} = \frac{l}{2}$$

$$\Rightarrow \frac{\ell}{2} - \frac{\ell}{4} = 0 + \frac{\ell}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \quad \because \cos \frac{n\pi}{2} = 0 \text{ if } n \text{ is odd}$$

$$\Rightarrow \frac{\ell}{4} = \frac{\ell}{\pi} \left[1 \sin \frac{\pi}{2} + \frac{1}{2} \sin \pi + \frac{1}{3} \sin \frac{3\pi}{2} + \dots \right]$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{1}(1) + \frac{1}{2}(0) + \frac{1}{3}(-1) + \dots \quad \because \sin \frac{\pi}{2} = 1; \sin \frac{3\pi}{2} = -1; \sin \frac{5\pi}{2} = 1; \sin \frac{7\pi}{2} = -1 \text{ etc..}$$

$$\boxed{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}}$$

i) Let $x = \ell$ be a point of continuity

$$(2) \Rightarrow f(l) = \frac{\ell}{4} + \frac{2\ell}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} \cos n\pi + \frac{\ell}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi \quad \because f(x) = \begin{cases} l-x, & 0 < x \leq l \\ 0, & l \leq x < 2l \end{cases}$$

$$0 = \frac{\ell}{4} + \frac{2\ell}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} (-1)^n + 0 \quad \Rightarrow f(l) = l - l = 0$$

$$-\frac{\ell}{4} = \frac{2\ell}{\pi^2} \left[-\frac{1}{1^2} + \frac{1}{3^2} - \frac{1}{5^2} + \dots \right]$$

$$-\frac{\pi^2}{8} = - \left[\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \dots \right]$$

$$\boxed{\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \dots = \frac{\pi^2}{8}}$$

3. Find the Fourier series expansion of $f(x) = \begin{cases} x, & 0 \leq x \leq 3 \\ 6-x, & 3 \leq x \leq 6 \end{cases}$ and hence find the value

of $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

Solution:

Given $f(x) = \begin{cases} x, & 0 \leq x \leq 3 \\ 6-x, & 3 \leq x \leq 6 \end{cases}$

General Fourier is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$

Here $\ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{6-0}{2} = 3 \quad \therefore \ell = 3$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3} \quad \text{----- (1)}$$

$$a_0 = \frac{1}{\ell} \int_0^{2\ell} f(x) dx = \frac{1}{3} \int_0^6 f(x) dx = \frac{1}{3} \left[\int_0^3 x dx + \int_3^6 (6-x) dx \right]$$

$$= \frac{1}{3} \left\{ \left[\frac{x^2}{2} \right]_0^3 + \left[\frac{(6-x)^2}{-2} \right]_3^6 \right\}$$

$$= \frac{1}{3} \left\{ \left[\left\{ \frac{3^2}{2} \right\} - \{0\} \right] + \left[\{0\} - \left\{ \frac{(3)^2}{-2} \right\} \right] \right\} = \frac{1}{3} \left[\frac{9}{2} + \frac{9}{2} \right] = \frac{9}{3}$$

$$\boxed{a_0 = 3}$$

$$a_n = \frac{1}{\ell} \int_0^{2\ell} f(x) \cos \frac{n\pi x}{\ell} dx$$

$$= \frac{1}{3} \int_0^6 f(x) \cos \frac{n\pi x}{3} dx$$

$$= \frac{1}{3} \left[\int_0^3 x \cos \frac{n\pi x}{3} dx + \int_3^6 (6-x) \cos \frac{n\pi x}{3} dx \right]$$

$$= \frac{1}{3} \left\{ \left[\cancel{\left(x \right) \left(\frac{\sin \frac{n\pi x}{3}}{\frac{n\pi}{3}} \right)} - (1) \left(\frac{-\cos \frac{n\pi x}{3}}{\frac{n^2 \pi^2}{9}} \right) \right]_0^3 + \left[\cancel{(6-x) \left(\frac{\sin \frac{n\pi x}{3}}{\frac{n\pi}{3}} \right)} - (-1) \left(\frac{-\cos \frac{n\pi x}{3}}{\frac{n^2 \pi^2}{9}} \right) \right]_3^6 \right\}$$

$$= \frac{1}{3} \left\{ \left[\frac{9}{n^2 \pi^2} \cos \frac{n\pi x}{3} \right]_0^3 - \left[\frac{9}{n^2 \pi^2} \cos \frac{n\pi x}{3} \right]_3^6 \right\}$$

$$= \frac{9}{3n^2 \pi^2} \left\{ \left[\cos \frac{n\pi x}{3} \right]_0^3 - \left[\cos \frac{n\pi x}{3} \right]_3^6 \right\}$$

$$= \frac{3}{n^2 \pi^2} \{ [\cos n\pi - \cos 0] - [\cos 2n\pi - \cos n\pi] \}$$

$$= \frac{3}{n^2 \pi^2} [\cos n\pi - 1 - 1 + \cos n\pi] = \frac{3}{n^2 \pi^2} [2(-1)^n - 2] = \frac{6}{n^2 \pi^2} [(-1)^n - 1]$$

$$= \frac{6}{n^2 \pi^2} \begin{cases} -2, & n=1,3,5,\dots \\ 0, & n=2,4,6,\dots \end{cases}$$

$$a_n = \begin{cases} \frac{-12}{n^2 \pi^2}, n=1,3,5,\dots \\ 0, n=2,4,6,\dots \end{cases}$$

$$\begin{aligned} b_n &= \frac{1}{\ell} \int_0^{2\ell} f(x) \sin \frac{n\pi x}{\ell} dx = \frac{1}{3} \int_0^6 f(x) \sin \frac{n\pi x}{3} dx = \frac{1}{3} \left[\int_0^3 x \sin \frac{n\pi x}{3} dx + \int_3^6 (6-x) \sin \frac{n\pi x}{3} dx \right] \\ &= \frac{1}{3} \left\{ \left[(x) \left(\frac{-\cos \frac{n\pi x}{3}}{\frac{n\pi}{3}} \right) - (1) \left(\frac{-\sin \frac{n\pi x}{3}}{\frac{n^2 \pi^2}{9}} \right) \right]_0^3 + \left[(6-x) \left(\frac{-\cos \frac{n\pi x}{3}}{\frac{n\pi}{3}} \right) - (-1) \left(\frac{-\sin \frac{n\pi x}{3}}{\frac{n^2 \pi^2}{9}} \right) \right]_3^6 \right\} \\ &= \frac{1}{3} \left\{ \left[\frac{-3}{n\pi} x \cos \frac{n\pi x}{3} \right]_0^3 + \left[\frac{-3}{n\pi} (6-x) \cos \frac{n\pi x}{3} \right]_3^6 \right\} \\ &= \frac{-3}{3n\pi} \{ [3 \cos n\pi - 0] + [0 - 3 \cos n\pi] \} = \frac{-3}{3n\pi} [3 \cos n\pi - 3 \cos n\pi] \\ \therefore b_n &= 0 \end{aligned}$$

Substitute a_0, a_n, b_n in (1)

$$f(x) = \frac{3}{2} + \sum_{n=1,3,5}^{\infty} \frac{-12}{n^2 \pi^2} \cos \frac{n\pi x}{3} + \sum_{n=1}^{\infty} 0 \sin \frac{n\pi x}{3}$$

$$\therefore f(x) = \frac{3}{2} - \frac{12}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{n\pi x}{3} \text{------(2)}$$

Deduction:

Let $x=0$ be a point of continuity

$$\begin{aligned} f(0) &= \frac{3}{2} - \frac{12}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} \cos 0 \quad f(x) = \begin{cases} x, & 0 \leq x \leq 3 \\ 6-x, & 3 \leq x \leq 6 \end{cases} \\ \Rightarrow 0 &= \frac{3}{2} - \frac{12}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} \\ \Rightarrow \frac{12}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} &= \frac{3}{2} \\ \therefore \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{8} \Rightarrow \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8} \right] \end{aligned}$$

4. Find the Fourier series for $f(x) = |x|$ in $-\pi < x < \pi$ and deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$.

Solution:

$$\text{Given } f(x) = |x|, \quad -\pi < x < \pi$$

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} \quad \therefore \ell = \pi$$

$$\text{Now, } f(-x) = |-x| = |x| = f(x)$$

$\therefore f(x)$ is an even function.

$$\therefore b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos nx \text{ ----- (1)} \quad \because l = \pi$$

To Find a_0 :

$$a_0 = \frac{2}{l} \int_0^l f(x) dx = \frac{2}{\pi} \int_0^{\pi} |x| dx = \frac{2}{\pi} \int_0^{\pi} x dx$$

$$= \frac{2}{\pi} \left(\frac{x^2}{2} \right)_0^{\pi} = \frac{2}{2\pi} (\pi^2 - 0) = \pi$$

$$\therefore \boxed{a_0 = \pi}$$

To Find a_n :

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx = \frac{2}{\pi} \int_0^{\pi} |x| \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx dx$$

$$= \frac{2}{\pi} \left[(x) \left(\frac{\sin nx}{n} \right) - (1) \left(\frac{-\cos nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(\frac{1}{n^2} \right) \cos nx \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left(\frac{1}{n^2} \right) [\cos n\pi - \cos 0]$$

$$= \frac{2}{\pi n^2} [(-1)^n - 1]$$

$$\boxed{a_n = \begin{cases} \frac{-4}{\pi n^2}, & \text{if } n=1,3,5,\dots \\ 0, & \text{if } n=2,4,6,\dots \end{cases}}$$

$$\therefore (1) \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1,3,5,\dots}^{\infty} \frac{-4}{n^2 \pi} \cos nx$$

$$\therefore \boxed{f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} \cos nx} \text{ ----- (2)}$$

Deduction:

Let $x=0$ be a point of continuity

$$(2) \Rightarrow f(0) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} \cos 0$$

$$\Rightarrow 0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2}$$

$$f(x) = |x| \Rightarrow f(0) = 0$$

$$\Rightarrow \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} = \frac{\pi}{2}$$

$$\sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

$$\therefore \boxed{1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}}$$

5. Find the Fourier series for $f(x) = x^2$ in $-\pi \leq x \leq \pi$ and deduce that

$$(a) \quad 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}.$$

$$(b) \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} = \frac{\pi^2}{12}.$$

$$(c) \quad \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$$

Solution:

Given $f(x) = x^2$, $-\pi \leq x \leq \pi$

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} = \pi \quad \therefore \ell = \pi$$

$$\text{Now, } f(-x) = x^2 = (-x)^2 = x^2 = f(x)$$

$\therefore f(x)$ is an even function.

$$\therefore b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{----- (1)} \quad \because \ell = \pi$$

To Find a_0 :

$$a_0 = \frac{2}{l} \int_0^l f(x) dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2}{3\pi} [\pi^3 - 0] = \frac{2\pi^2}{3}$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

To find a_n :

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx$$

$$= \frac{2}{\pi} \left[(x^2) \left(\frac{\sin nx}{n} \right) - (2x) \left(\frac{-\cos nx}{n^2} \right) + (2) \left(\frac{\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(\frac{2}{n^2} \right) x \cos nx \right]_0^{\pi}$$

$$= \frac{4}{\pi n^2} (\pi \cos n\pi - 0)$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$$\therefore (1) \Rightarrow f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

$$\therefore \boxed{f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx} \text{-----} (2)$$

Deduction: a

Let $x = \pi$ be a point of continuity

$$\begin{aligned} (2) \Rightarrow f(\pi) &= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \pi \\ \Rightarrow \pi^2 &= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n^2} & \because f(x) = x^2 \Rightarrow f(\pi) = \pi^2 \\ \Rightarrow \pi^2 - \frac{\pi^2}{3} &= 4 \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} \\ \Rightarrow \frac{2\pi^2}{3} &= 4 \sum_{n=1}^{\infty} \frac{1}{n^2} & \because (-1)^{2n} = 1 \\ \therefore \boxed{1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}} \end{aligned}$$

Deduction: b

Let $x = 0$ be a point of continuity

$$\begin{aligned} (2) \Rightarrow f(0) &= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos 0 \\ \Rightarrow 0 &= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} & \because f(x) = x^2 \Rightarrow f(0) = 0 \\ \Rightarrow -\frac{\pi^2}{3} &= 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \\ \Rightarrow -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} &= -\frac{\pi^2}{12} \\ \Rightarrow \boxed{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} = \frac{\pi^2}{12}} \end{aligned}$$

Deduction: c

By Parseval's identity for Fourier series,

$$\begin{aligned} \frac{2}{l} \int_0^l [f(x)]^2 dx &= \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 \\ \frac{2}{\pi} \int_0^{\pi} [x^2]^2 dx &= \frac{4\pi^4}{2} + \sum_{n=1}^{\infty} \left(\frac{4(-1)^n}{n^2} \right)^2 \\ \frac{2}{\pi} \int_0^{\pi} x^4 dx &= \frac{2\pi^4}{9} + 16 \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^4} \\ \frac{2}{\pi} \left[\left(\frac{\pi^5}{5} \right) - (0) \right] &= \frac{2\pi^4}{9} + 16 \sum_{n=1}^{\infty} \frac{1}{n^4} & \because (-1)^{2n} = 1 \\ \frac{2\pi^5}{5\pi} &= \frac{2\pi^4}{9} + 16 \sum_{n=1}^{\infty} \frac{1}{n^4} \end{aligned}$$

$$\frac{2\pi^4}{5} - \frac{2\pi^4}{9} = 16 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$2\pi^4 \left(\frac{1}{5} - \frac{1}{9} \right) = 16 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$2\pi^4 \times \frac{4}{45} = 16 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{8\pi^4}{45 \times 16} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$$

Obtain the Fourier series of $f(x) = x + x^2$ in $-\pi < x < \pi$. Hence show that

6.

$$i) \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad ii) \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$$

Solution:

Given $f(x) = x + x^2$ in $-\pi \leq x \leq \pi$

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} = \pi \quad \therefore \ell = \pi$$

$$\text{Now, } f(-x) = -x + (-x)^2$$

$$= -x + x^2 \neq f(x)$$

$$= -(x - x^2) \neq -f(x)$$

$$\therefore f(-x) \neq f(x) \quad \& \quad f(-x) \neq -f(x)$$

$\therefore f(x)$ is Neither even Nor odd function.

$$\text{General Fourier is } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{----- (1) } \quad \because \ell = \pi$$

To Find a_0 :

$$\begin{aligned} a_0 &= \frac{1}{\ell} \int_{-l}^l f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) dx = \frac{1}{\pi} \left[\frac{x^2}{2} + \frac{x^3}{3} \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left\{ \left(\frac{\pi^2}{2} + \frac{\pi^3}{3} \right) - \left(\frac{\pi^2}{2} - \frac{\pi^3}{3} \right) \right\} = \frac{1}{\pi} \left\{ \frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3} \right\} = \frac{1}{\pi} \left\{ \frac{2\pi^3}{3} \right\} = \frac{2\pi^2}{3} \end{aligned}$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

To Find a_n :

$$a_n = \frac{1}{\ell} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \cos nx dx$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \left[\cancel{\left(x + x^2 \right) \left(\frac{\sin nx}{n} \right)} - (1 + 2x) \left(\frac{-\cos nx}{n^2} \right) + (2) \left(\frac{\sin nx}{n^3} \right) \right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi} \left[\left(\frac{1}{n^2} \right) (1 + 2x) \cos nx \right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi n^2} \left\{ \left[(1 + 2\pi) \cos n\pi \right] - \left[(1 - 2\pi) \cos(-n\pi) \right] \right\} \\
&= \frac{1}{\pi n^2} \left[(-1)^n (\cancel{1} + 2\pi - \cancel{1} + 2\pi) \right] \quad \because \cos(-n\pi) = \cos n\pi = (-1)^n \\
&= \frac{(-1)^n}{n^2 \pi} [4\pi] = \frac{4(-1)^n}{n^2}
\end{aligned}$$

$$a_n = \frac{4(-1)^n}{n^2}$$

To Find b_n :

$$\begin{aligned}
b_n &= \frac{1}{\ell} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \sin nx dx \\
&= \frac{1}{\pi} \left[\left(x + x^2 \right) \left(\frac{-\cos nx}{n} \right) - \cancel{(1 + 2x) \left(\frac{-\sin nx}{n^2} \right)} + (2) \left(\frac{\cos nx}{n^3} \right) \right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi} \left[\left(\frac{-1}{n} \right) (x + x^2) \cos nx + \left(\frac{2}{n^3} \right) \cos nx \right]_{-\pi}^{\pi} \quad \because \cos(-n\pi) = \cos n\pi = (-1)^n \\
&= \frac{1}{\pi} \left\{ \left[\left(\frac{-1}{n} \right) (\pi + \pi^2) \cos n\pi + \left(\frac{2}{n^3} \right) \cos n\pi \right] - \left[\left(\frac{-1}{n} \right) (-\pi + \pi^2) \cos n\pi + \left(\frac{2}{n^3} \right) \cos(-n\pi) \right] \right\} \\
&= \frac{(-1)^n}{\pi} \left(-\frac{\pi}{n} - \cancel{\frac{\pi^2}{n}} + \cancel{\frac{2}{n^3}} - \frac{\pi}{n} + \cancel{\frac{\pi^2}{n}} - \cancel{\frac{2}{n^3}} \right) = -\frac{2\pi(-1)^n}{n\pi} \\
\therefore b_n &= \frac{-2}{n} (-1)^n
\end{aligned}$$

$$(1) \Rightarrow f(x) = \frac{\pi^2}{3} + \sum_1^{\infty} \frac{4(-1)^n}{n^2} \cos nx + \sum_1^{\infty} \frac{-2(-1)^n}{n} \sin nx$$

$$f(x) = \frac{\pi^2}{3} + 4 \sum_1^{\infty} \frac{(-1)^n}{n^2} \cos nx - 2 \sum_1^{\infty} \frac{(-1)^n}{n} \sin nx \quad \text{----- (2)}$$

Deduction: 1

Let $x = \pi$ be the point of discontinuity

$$(2) \Rightarrow f(\pi) = \frac{\pi^2}{3} + 4 \sum_1^{\infty} \frac{(-1)^n}{n^2} \cos n\pi - 2 \sum_1^{\infty} \frac{(-1)^n}{n} \sin n\pi$$

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_1^{\infty} \frac{1}{n^2} \quad \because f(x) = x + x^2 \Rightarrow f(\pi) = \frac{f(-\pi) + f(\pi)}{2} = \frac{-\pi + \pi^2 + \pi + \pi^2}{2} = \frac{2\pi^2}{2} = \pi^2$$

$$\pi^2 - \frac{\pi^2}{3} = 4 \sum_1^{\infty} \frac{1}{n^2}$$

$$\frac{2\pi^2}{3 \times 4} = \sum_1^{\infty} \frac{1}{n^2}$$

$$\boxed{\sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

Deduction: 2

Let $x=0$ be the point of continuity

$$(2) \Rightarrow f(0) = \frac{\pi^2}{3} + 4 \sum_1^{\infty} \frac{(-1)^n}{n^2} \cos 0 - 2 \sum_1^{\infty} \frac{(-1)^n}{n} \sin 0$$

$$0 = \frac{\pi^2}{3} + 4 \sum_1^{\infty} \frac{(-1)^n}{n^2} \quad f(x) = x + x^2 \Rightarrow f(0) = 0$$

$$-\frac{\pi^2}{3} = 4 \left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} \dots \right]$$

$$\boxed{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}}$$

7. Find the Fourier series expansion of $f(x)$ where $f(x) = \begin{cases} \pi + x, & -\pi \leq x \leq 0 \\ \pi - x, & 0 \leq x \leq \pi \end{cases}$ and hence deduce that

$$\sum_1^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Solution:

$$\text{Given } f(x) = \begin{cases} \pi + x, & -\pi \leq x \leq 0 \\ \pi - x, & 0 \leq x \leq \pi \end{cases}$$

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} = \pi \quad \therefore \ell = \pi$$

$$\text{Let } f(x) = \begin{cases} f_1(x), & -\pi \leq x \leq 0 \\ f_2(x), & 0 \leq x \leq \pi \end{cases}$$

Where

$$f_1(x) = \pi + x \quad f_2(x) = \pi - x$$

$$f_1(-x) = \pi - x = f_2(x) \quad f_2(-x) = \pi + x = f_1(x)$$

$\therefore f(x)$ is an even function.

$$\therefore \boxed{b_n = 0}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$$

$$\therefore \boxed{f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos nx} \text{ ----- (1) } \quad \because \ell = \pi$$

To Find a_0 :

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (\pi - x) dx = \frac{2}{\pi} \int_0^{\pi} x dx$$

$$= \frac{2}{\pi} \left(\pi x - \frac{x^2}{2} \right)_0^{\pi} = \frac{2}{\pi} \left(\pi^2 - \frac{\pi^2}{2} \right) = \frac{2}{\pi} \times \frac{\pi^2}{2} = \pi$$

$$\therefore \boxed{a_0 = \pi}$$

To Find a_n :

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi - x) \cos nx dx$$

$$= \frac{2}{\pi} \left[(\pi - x) \left(\frac{\sin nx}{n} \right) - (-1) \left(\frac{-\cos nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(\frac{-1}{n^2} \right) \cos nx \right]_0^{\pi}$$

$$= \frac{-2}{n^2 \pi} [\cos n\pi - \cos 0]$$

$$= \frac{-2}{\pi n^2} [(-1)^n - 1]$$

$$\boxed{a_n = \begin{cases} \frac{4}{\pi n^2}, & \text{if } n=1,3,5,\dots \\ 0, & \text{if } n=2,4,6,\dots \end{cases}}$$

$$\therefore (1) \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1,3,5,\dots}^{\infty} \frac{4}{n^2 \pi} \cos nx$$

$$\therefore \boxed{f(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} \cos nx} \text{----- (2)}$$

Deduction:

Let $x = 0$ be a point of continuity

$$(2) \Rightarrow f(0) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} \cos 0$$

$$\Rightarrow \pi = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} = \pi - \frac{\pi}{2}$$

$$\sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} = \frac{\pi}{2} \times \frac{\pi}{4} = \frac{\pi^2}{8}$$

$$\therefore \boxed{\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}}$$

$$\therefore f(x) = \begin{cases} \pi + x, & -\pi \leq x \leq 0 \\ \pi - x, & 0 \leq x \leq \pi \end{cases} \Rightarrow f(0) = \pi$$

$$\therefore \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{2n-1}$$

8. Determine the Fourier series expansion of $f(x)$ where $f(x) = \begin{cases} -1+x, & -\pi \leq x \leq 0 \\ 1+x, & 0 \leq x \leq \pi \end{cases}$ with $f(x+2\pi) = f(x)$.

Solution:

$$\text{Given } f(x) = \begin{cases} -1+x, & -\pi \leq x \leq 0 \\ 1+x, & 0 \leq x \leq \pi \end{cases}$$

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} = \pi \quad \therefore \ell = \pi$$

$$\text{Let } f(x) = \begin{cases} f_1(x), & -\pi \leq x \leq 0 \\ f_2(x), & 0 \leq x \leq \pi \end{cases}$$

Where

$$f_1(x) = -1+x$$

$$f_2(x) = 1+x$$

$$f_1(-x) = -1-x = -(1+x) = -f_2(x)$$

$$f_2(-x) = 1-x = -(-1+x) = -f_1(x)$$

$\therefore f(x)$ is an odd function.

$$\therefore \boxed{a_0 = a_n = 0}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{----- (1)} \quad \because \ell = \pi$$

To Find b_n :

$$b_n = \frac{1}{\ell} \int_{-l}^l f(x) \sin \frac{n\pi x}{\ell} dx = \frac{2}{\pi} \int_0^{\pi} (1+x) \sin nx dx$$

$$= \frac{2}{\pi} \left[(1+x) \left(\frac{-\cos nx}{n} \right) - (1) \left(\frac{-\sin nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(\frac{-1}{n} \right) (1+x) \cos nx \right]_0^{\pi}$$

$$= \frac{-2}{n\pi} [(1+\pi) \cos n\pi - (1+0) \cos 0]$$

$$= \frac{-2}{n\pi} [(1+\pi)(-1)^n - 1]$$

$$\boxed{b_n = \frac{2}{n\pi} [1 - (1+\pi)(-1)^n]}$$

$$\therefore (1) \Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} [1 - (1+\pi)(-1)^n] \sin nx$$

$$\therefore \boxed{f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} [1 - (1+\pi)(-1)^n] \sin nx}$$

9. Find the Fourier series for $f(x)$ where $f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$.

Solution:

$$\text{Given } f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$$

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{1 - (-1)}{2} = \frac{2}{2} = 1 \quad \therefore \ell = 1$$

$$\text{Let } f(x) = \begin{cases} f_1(x), & -1 \leq x \leq 0 \\ f_2(x), & 0 \leq x \leq 1 \end{cases}$$

Where

$$f_1(x) = 0 \qquad f_2(x) = 1$$

$$f_1(-x) = 0 \neq f_2(x) \qquad f_2(-x) = 1 \neq f_1(x)$$

$$f_1(-x) = 0 \neq f_2(x) \qquad f_2(-x) = 1 \neq f_1(x)$$

$\therefore f(x)$ is Neither even Nor odd function.

$$\text{General Fourier is } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

$$\therefore \left[f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \right] \text{----- (1) } \because l = 1$$

To Find a_0 :

$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx = \frac{1}{1} \int_{-1}^1 f(x) dx = \int_{-1}^0 0 dx + \int_0^1 1 dx = [x]_0^1 = 1 - 0 = 1$$

$$\boxed{\therefore a_0 = 1}$$

To Find a_n :

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx$$

$$= \frac{1}{1} \int_{-1}^1 f(x) \cos \frac{n\pi x}{1} dx$$

$$= \int_{-1}^0 0 dx + \int_0^1 1 \cos n\pi x dx$$

$$= \left[\frac{\sin n\pi x}{n\pi} \right]_0^1$$

$$= [\sin n\pi - \sin 0] = 0$$

$$\boxed{a_n = 0}$$

To Find b_n :

$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{1}{1} \int_{-1}^1 f(x) \sin \frac{n\pi x}{1} dx$$

$$= \int_{-1}^0 0 dx + \int_0^1 1 \sin n\pi x dx$$

$$= \left[\frac{-\cos n\pi x}{n\pi} \right]_0^1$$

$$\begin{aligned}
&= \frac{-1}{n\pi} [\cos n\pi x]_0^1 \\
&= \frac{-1}{n\pi} [\cos n\pi - \cos 0]_0^1 \\
&= \frac{1}{n\pi} [1 - (-1)^n]
\end{aligned}$$

$$b_n = \begin{cases} \frac{2}{n\pi}, & \text{if } n = 1, 3, 5, \dots \\ 0, & \text{if } n = 2, 4, 6, \dots \end{cases}$$

$$(1) \Rightarrow f(x) = \frac{1}{2} + \sum_{n=1,3}^{\infty} \frac{2}{n\pi} \sin n\pi x.$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1,3}^{\infty} \frac{1}{n} \sin n\pi x$$

10. Find the Fourier series for $f(x) = |\cos x|$ in the interval $(-\pi, \pi)$.

Solution:

$$\text{Given } f(x) = |\cos x|, \quad -\pi < x < \pi$$

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} \quad \therefore \ell = \pi$$

$$\text{Now, } f(-x) = |\cos(-x)| = |\cos x| = f(x)$$

$\therefore f(x)$ is an even function.

$$\therefore \boxed{b_n = 0}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{----- (1)} \quad \because \ell = \pi$$

To find a_0 :

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} |\cos x| dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} \cos x dx + \int_{\pi/2}^{\pi} -\cos x dx \right] \quad \because |\cos x| = \begin{cases} \cos x, & 0 < x < \frac{\pi}{2} \\ -\cos x, & \frac{\pi}{2} < x < \pi \end{cases}$$

$$= \frac{2}{\pi} \left[(\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi} \right]$$

$$= \frac{2}{\pi} \left[\left(\sin \frac{\pi}{2} - \sin 0 \right) - \left(\sin \pi - \sin \frac{\pi}{2} \right) \right] = \frac{2}{\pi} (1 - (-1))$$

$$\therefore \boxed{a_0 = \frac{4}{\pi}}$$

To find a_n :

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx = \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx$$

$$a_n = \frac{2}{\pi} \int_0^\pi |\cos x| \cos nx dx$$

$$a_n = \frac{2}{\pi} \left[\int_0^{\pi/2} \cos x \cos nx dx + \int_{\pi/2}^\pi -\cos x \cos nx dx \right]$$

$$a_n = \frac{2}{\pi} \left[\int_0^{\pi/2} \cos nx \cos x dx - \int_{\pi/2}^\pi \cos nx \cos x dx \right]$$

$$\cos A \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)] \quad \text{Here } A = nx \quad B = x$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \left[\int_0^{\pi/2} \frac{1}{2} [\cos(n+1)x + \cos(n-1)x] dx - \int_{\pi/2}^\pi \frac{1}{2} [\cos(n+1)x + \cos(n-1)x] dx \right] \\ &= \frac{1}{\pi} \left[\int_0^{\pi/2} \frac{1}{2} [\cos(n+1)x + \cos(n-1)x] dx - \int_{\pi/2}^\pi [\cos(n+1)x + \cos(n-1)x] dx \right] \\ &= \frac{1}{\pi} \left\{ \left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right]_0^{\pi/2} - \left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right]_{\pi/2}^\pi \right\} \\ &= \frac{1}{\pi} \left[\left(\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1} \right) - (0) \right] - \frac{1}{\pi} \left[(0) - \left(\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1} \right) \right] \\ &= \frac{1}{\pi} \left(\frac{\sin\left(\frac{n\pi}{2} + \frac{\pi}{2}\right)}{n+1} + \frac{\sin\left(\frac{n\pi}{2} - \frac{\pi}{2}\right)}{n-1} + \frac{\sin\left(\frac{n\pi}{2} + \frac{\pi}{2}\right)}{n+1} + \frac{\sin\left(\frac{n\pi}{2} - \frac{\pi}{2}\right)}{n-1} \right) \end{aligned}$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin\left(\frac{n\pi}{2} + \frac{\pi}{2}\right) = \sin \frac{n\pi}{2} \cos \frac{\pi}{2} + \cos \frac{n\pi}{2} \sin \frac{\pi}{2} = \cos \frac{n\pi}{2} \quad \because \cos \frac{\pi}{2} = 0 \& \sin \frac{\pi}{2} = 1$$

$$\sin\left(\frac{n\pi}{2} - \frac{\pi}{2}\right) = \sin \frac{n\pi}{2} \cos \frac{\pi}{2} - \cos \frac{n\pi}{2} \sin \frac{\pi}{2} = -\cos \frac{n\pi}{2} \quad \because \cos \frac{\pi}{2} = 0 \& \sin \frac{\pi}{2} = 1$$

$$a_n = \frac{1}{\pi} \left(\frac{\cos \frac{n\pi}{2}}{n+1} - \frac{\cos \frac{n\pi}{2}}{n-1} + \frac{\cos \frac{n\pi}{2}}{n+1} - \frac{\cos \frac{n\pi}{2}}{n-1} \right)$$

$$= \frac{2}{\pi} \cos \frac{n\pi}{2} \left(\frac{1}{n+1} - \frac{1}{n-1} \right)$$

$$= \frac{2}{\pi} \cos \frac{n\pi}{2} \left(\frac{n-1-n-1}{(n+1)(n-1)} \right)$$

$$a_n = \frac{2}{\pi} \cos \frac{n\pi}{2} \left(\frac{-2}{(n^2-1)} \right)$$

$$a_n = \frac{-4}{\pi(n^2-1)} \cos \frac{n\pi}{2}, \text{ Provided } n \neq 1$$

When $n=1$

$$\begin{aligned} a_1 &= \frac{2}{\pi} \int_0^\pi |\cos x| \cos x dx \\ &= \frac{2}{\pi} \left[\int_0^{\pi/2} \cos x \cos x dx + \int_{\pi/2}^\pi -\cos x \cos x dx \right] \\ &= \frac{2}{\pi} \left[\int_0^{\pi/2} \cos^2 x dx - \int_{\pi/2}^\pi \cos^2 x dx \right] \\ &= \frac{2}{\pi} \left[\int_0^{\pi/2} \left(\frac{1+\cos 2x}{2} \right) dx - \int_{\pi/2}^\pi \left(\frac{1+\cos 2x}{2} \right) dx \right] \\ &= \frac{2}{2\pi} \left[\int_0^{\pi/2} (1+\cos 2x) dx - \int_{\pi/2}^\pi (1+\cos 2x) dx \right] \\ &= \frac{1}{\pi} \left[\left(x + \frac{\sin 2x}{2} \right)_0^{\pi/2} - \left(x + \frac{\sin 2x}{2} \right)_{\pi/2}^\pi \right] \\ &= \frac{1}{\pi} \left\{ \left[\left(\frac{\pi}{2} + \frac{\sin \pi}{2} \right) - (0) \right] - \left[\left(\pi + \frac{\sin 2\pi}{2} \right) - \left(\frac{\pi}{2} + \frac{\sin \pi}{2} \right) \right] \right\} \\ a_1 &= \frac{1}{\pi} \left[\frac{\pi}{2} - \pi + \frac{\pi}{2} \right] \end{aligned}$$

$$a_1 = 0$$

$$\begin{aligned} \therefore f(x) &= \frac{\pi}{2} + \sum_1^\infty \frac{-4}{\pi(n^2-1)} \cos \frac{n\pi}{2} \cos nx \\ \therefore f(x) &= \frac{\pi}{2} - \frac{4}{\pi} \sum_1^\infty \frac{1}{(n^2-1)} \cos \frac{n\pi}{2} \cos nx \end{aligned}$$

11. Find the half range Fourier sine series for $f(x) = x(\pi - x)$ in the interval $(0, \pi)$ and deduce that

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots \infty$$

Solution:

Given $f(x) = x(\pi - x) = \pi x - x^2, \quad (0, \pi)$

: General Fourier is $f(x) = \sum_{n=1}^\infty b_n \sin \frac{n\pi x}{\ell}$

Here $\ell = \text{Upper Limit} - \text{Lower Limit} = \pi - 0 = \pi \quad \therefore \ell = \pi$

$$f(x) = \sum_{n=1}^\infty b_n \sin nx \quad \text{----- (1)}$$

To Find b_n :

$$\begin{aligned} b_n &= \frac{2}{\ell} \int_0^\ell f(x) \sin \frac{n\pi x}{\ell} dx \\ &= \frac{2}{\ell} \int_0^\ell x(\pi - x) \sin nx dx \end{aligned}$$

$$= \frac{2}{\ell} \int_0^{\ell} (\pi x - x^2) \sin nx \, dx$$

$$b_n = \frac{2}{\pi} \left[\cancel{(\pi x - x^2) \left(\frac{-\cos nx}{n} \right)} - \cancel{(\pi - 2x) \left(\frac{-\sin nx}{n^2} \right)} + (-2) \left(\frac{\cos nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[-\frac{2}{n^3} \cos nx \right]_0^{\pi}$$

$$= \frac{-4}{\pi n^3} [\cos n\pi - \cos 0]$$

$$= \frac{-4}{\pi n^3} [(-1)^n - 1]$$

$$b_n = \begin{cases} \frac{8}{\pi n^3}, & \text{if } n=1,3,5,\dots \\ 0, & \text{if } n=2,4,6,\dots \end{cases}$$

The required Fourier sine series be

$$(1) \Rightarrow f(x) = \sum_{n=1,3,5,\dots}^{\infty} \frac{8}{\pi n^3} \sin nx$$

$$f(x) = \frac{8}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} \sin nx \quad \text{-----} (2)$$

Deduction:

Let $x = \frac{\pi}{2}$ be a point of continuity.

$$(2) \Rightarrow f\left(\frac{\pi}{2}\right) = \frac{8}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} \sin\left(\frac{n\pi}{2}\right) \quad \because f(x) = x(\pi - x) \Rightarrow f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{2} - \frac{\pi^2}{4} = \frac{2\pi^2 - \pi^2}{4}$$

$$\Rightarrow \frac{\pi^2}{4} = \frac{8}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} \sin\left(\frac{n\pi}{2}\right) \quad \Rightarrow f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$$

$$\Rightarrow \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} \sin\left(\frac{n\pi}{2}\right) = \frac{\pi^3}{32}$$

$$\Rightarrow \frac{1}{1^3} \sin \frac{\pi}{2} + \frac{1}{3^3} \sin \frac{\pi}{2} + \frac{1}{5^3} \sin \frac{\pi}{2} + \dots = \frac{\pi^3}{32}$$

$$\Rightarrow \boxed{\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots = \frac{\pi^3}{32}}$$

12. Find the half – range cosine series for $f(x) = (x-1)^2$ in $(0, 1)$. Hence show that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$.

Solution:

Given:

Here ℓ = Upper Limit – Lower Limit = $1-0=1 \quad \therefore \ell=1$

Let the cosine series be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \quad \because l=1 \text{-----} (1)$$

To Find a_0 :

$$a_0 = \frac{2}{\ell} \int_0^{\ell} f(x) dx = \frac{2}{1} \int_0^1 (x^2 - 2x + 1) dx = 2 \left[\frac{x^3}{3} - 2 \frac{x^2}{2} + x \right]_0^1 = 2 \left(\frac{1}{3} - 1 + 1 \right) = \frac{2}{3}$$

$$\boxed{a_0 = \frac{2}{3}}$$

To Find a_n :

$$a_n = 2 \int_0^1 (x^2 - 2x + 1) \cos n\pi x dx$$

$$= 2 \left[\cancel{(x^2 - 2x + 1)} \left(-\frac{\sin n\pi x}{n\pi} \right) - (2x - 2) \left(\frac{-\cos n\pi x}{n^2 \pi^2} \right) + (2) \left(\frac{\sin n\pi x}{n^3 \pi^3} \right) \right]_0^1$$

$$= 2 \left[\left(\frac{1}{n^2 \pi^2} \right) (2x - 2) \cos n\pi x \right]_0^1$$

$$= \frac{2}{n^2 \pi^2} [(0) - (-2 \cos 0)]$$

$$a_n = \frac{4}{n^2 \pi^2}$$

$$(1) \Rightarrow f(x) = \frac{2/3}{2} + \sum_1^{\infty} \frac{4}{n^2 \pi^2} \cos n\pi x$$

$$\therefore \boxed{f(x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_1^{\infty} \frac{1}{n^2} \cos n\pi x} \text{-----} (2)$$

Deduction:

Let $x=0$ be a point of discontinuity

$$(2) \Rightarrow f(0) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos 0$$

$$\Rightarrow \frac{1}{2} = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\because f(x) = (x-1)^2 \Rightarrow f(0) = \frac{f(0) + f(1)}{2} = \frac{1+0}{2} = \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} - \frac{1}{3} = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{1}{6} \times \frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}}$$

13. Find the Fourier cosine series for $x(\pi - x)$ in $0 < x < \pi$. Hence show that $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$.

Solution:

Given:

$$\text{Let } f(x) = x(\pi - x), \quad 0 < x < \pi$$

$$f(x) = \pi x - x^2$$

Here $\ell = \text{Upper Limit} - \text{Lower Limit} = \pi - 0 = \pi \quad \therefore \ell = \pi$

Let the cosine series be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \because l = \pi \text{ --- (1)}$$

To Find a_0 :

$$\begin{aligned} a_0 &= \frac{2}{\ell} \int_0^{\ell} f(x) dx = \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) dx = \frac{2}{\pi} \left[\frac{\pi x^2}{2} - \frac{x^3}{3} \right]_0^{\pi} = \frac{2}{\pi} \left[\left(\frac{\pi^3}{2} - \frac{\pi^3}{3} \right) - (0) \right] \\ &= \frac{2}{\pi} \left[\frac{3\pi^3 - 2\pi^3}{6} \right] = \frac{2}{\pi} \left[\frac{\pi^3}{6} \right] = \frac{\pi^2}{3} \end{aligned}$$

$$\boxed{a_0 = \frac{\pi^2}{3}}$$

To Find a_n :

$$a_n = 2 \int_0^{\pi} (x^2 - 2x + 1) \cos n\pi x dx$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \cos nx dx \\ &= \frac{2}{\pi} \left[(\pi x - x^2) \left(\frac{\sin nx}{n} \right) - (\pi - 2x) \left(\frac{-\cos nx}{n^2} \right) + (-2) \left(\frac{-\sin nx}{n^3} \right) \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[\left(\frac{1}{n^2} \right) (\pi - 2x) \cos nx \right]_0^{\pi} \\ &= \frac{2}{\pi n^2} [-\pi \cos n\pi - \pi \cos 0] = \frac{-2\pi}{\pi n^2} [(-1)^n + 1] \\ &= \frac{-2}{n^2} [(-1)^n + 1] \end{aligned}$$

$$\boxed{a_n = \begin{cases} -\frac{4}{n^2}, & \text{if } n=2,4,6,\dots \\ 0, & \text{if } n=1,3,5,\dots \end{cases}}$$

$$\therefore \boxed{f(x) = \frac{\pi^2}{6} + \sum_{n=2,4,\dots}^{\infty} -\frac{4}{n^2} \cos nx.}$$

Deduction:

Let the Parseval's identity for Fourier cosine series be

$$\begin{aligned} \frac{2}{l} \int_0^l [f(x)]^2 dx &= \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 \\ \frac{2}{\pi} \int_0^{\pi} [\pi x - x^2]^2 dx &= \frac{\pi^4}{2} + \sum_{n=2,4,6,\dots}^{\infty} \left(\frac{-4}{n^2} \right)^2 \\ \frac{2}{\pi} \int_0^{\pi} [\pi^2 x^2 - 2\pi x^3 + x^4] dx &= \frac{\pi^4}{18} + 16 \sum_{n=2,4,6,\dots}^{\infty} \frac{1}{n^4} \end{aligned}$$

$$\frac{2}{\pi} \left[\left(\frac{\pi^5}{3} - \frac{\pi^5}{2} + \frac{\pi^5}{5} \right) - (0) \right] = \frac{\pi^4}{18} + 16 \sum_{n=1}^{\infty} \frac{1}{(2n)^4} \quad \because \sum_{n=2,4,6,\dots}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{2n}$$

$$\frac{2\pi^5}{\pi} \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) = \frac{\pi^4}{18} + \frac{16}{16} \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$2\pi^4 \left(\frac{10-15+6}{30} \right) - \frac{\pi^4}{18} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{\pi^4}{15} - \frac{\pi^4}{18} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{6\pi^4 - 5\pi^4}{90}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}}$$

Find the cosine series for $f(x) = x$ in $(0, \pi)$ and then using Parseval's theorem, show that

14.

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96}.$$

Solution:

Given:

Let $f(x) = x$, $0 < x < \pi$

Here $\ell = \text{Upper Limit} - \text{Lower Limit} = \pi - 0 = \pi \quad \therefore \ell = \pi$

Let the cosine series be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi \quad \because \ell = \pi$$

To Find a_0 :

$$a_0 = \frac{2}{\ell} \int_0^{\ell} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x dx = \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = \frac{2}{2\pi} [\pi^2 - 0] = \pi$$

$$\boxed{a_0 = \pi}$$

To Find a_n :

$$a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx dx$$

$$= \frac{2}{\pi} \left[(x) \left(\frac{\sin nx}{n} \right) - (1) \left(\frac{-\cos nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(\frac{1}{n^2} \right) \cos nx \right]_0^{\pi}$$

$$= \frac{2}{\pi n^2} [\cos n\pi - \cos 0]$$

$$= \frac{2}{n^2 \pi} [(-1)^n - 1]$$

$$a_n = \begin{cases} \frac{-4}{n^2\pi}, & \text{if } n = 1, 3, 5, \dots \\ 0, & \text{if } n = 2, 4, 6, \dots \end{cases}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1,3,5,\dots}^{\infty} \frac{-4}{\pi n^2} \cos nx$$

Deduction:

Let the Parseval's identity for Fourier cosine series be

$$\frac{2}{l} \int_0^l [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{\pi^2}{2} + \sum_{n=1,3,5,\dots}^{\infty} \left(\frac{-4}{n^2\pi} \right)^2$$

$$\frac{2}{\pi} \left(\frac{x^3}{3} \right)_0^{\pi} = \frac{\pi^2}{2} + \sum_{n=1,3,5,\dots}^{\infty} \frac{16}{\pi^2 n^4}$$

$$\frac{2}{3\pi} (\pi^3 - 0) = \frac{\pi^2}{2} + \frac{16}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^4}$$

$$\frac{2\pi^2}{3} - \frac{\pi^2}{2} = \frac{16}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^4}$$

$$\frac{\pi^2}{6} \times \frac{\pi^2}{16} = \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^4}$$

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots \infty = \frac{\pi^4}{96}$$

15. Find the complex form of the Fourier series of $f(x) = \cos ax$ in $(-\pi, \pi)$, where a is not an integer.

Solution:

Given $f(x) = \cos ax$ in $(-\pi, \pi)$

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} \quad \therefore \ell = \pi$$

Let the complex form of the Fourier series be

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in x} \quad \because l = \pi$$

$$C_n = \frac{1}{2l} \int_{-l}^l f(x) e^{-\frac{in\pi x}{l}} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos ax e^{-inx} dx$$

$$\therefore \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx] \quad \text{Here } a = -in \text{ \& } b = a$$

$$= \frac{1}{2\pi} \left\{ \frac{e^{-inx}}{((in)^2 - a^2)} (-in \cos ax + a \sin ax) \right\}_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(a^2 - n^2)} \left[e^{-in\pi} (-in \cos a\pi + a \sin a\pi) - e^{in\pi} (-in \cos a\pi - a \sin a\pi) \right]$$

$$e^{in\pi} = \cos n\pi + i \sin n\pi = \cos n\pi = (-1)^n \quad \text{and} \quad e^{-in\pi} = \cos n\pi - i \sin n\pi = \cos n\pi = (-1)^n \quad \because \sin n\pi = 0$$

$$= \frac{1}{2\pi(a^2 - n^2)} \left[(-1)^n [-in \cos a\pi + a \sin a\pi + in \cos a\pi + a \sin a\pi] \right]$$

$$= \frac{1}{2\pi(a^2 - n^2)} \left[(-1)^n (2a \sin a\pi) \right]$$

$$C_n = \frac{(-1)^n a \sin a\pi}{\pi(a^2 - n^2)}$$

$$\therefore f(x) = \frac{a \sin a\pi}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n}{(a^2 - n^2)} e^{inx}$$

16. Find complex form of the Fourier series of the function $f(x) = e^{-x}$, $-1 < x < 1$

Solution:

Given $f(x) = e^{-x}$, $-1 < x < 1$

Here $\ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{1 - (-1)}{2} = \frac{2}{2} = 1 \quad \therefore \ell = 1$

Let the complex form of the Fourier series be

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x} \quad \because l = 1$$

$$C_n = \frac{1}{2l} \int_{-l}^l f(x) e^{-\frac{in\pi x}{l}} dx = \frac{1}{2} \int_{-1}^1 e^{-x} e^{-in\pi x} dx$$

$$C_n = \frac{1}{2} \int_{-1}^1 e^{-x - in\pi x} dx$$

$$= \frac{1}{2} \int_{-1}^1 e^{-(1+in\pi)x} dx$$

$$= \frac{1}{2} \left\{ \frac{e^{-(1+in\pi)x}}{-(1+in\pi)} \right\}_{-1}^1$$

$$= -\frac{1}{2(1+in\pi)} \left\{ e^{-(1+in\pi)} - e^{(1+in\pi)} \right\}$$

$$= -\frac{1}{2(1+in\pi)} \left\{ e^{-1} e^{-in\pi} - e^1 e^{in\pi} \right\}$$

$$e^{in\pi} = \cos n\pi + i \sin n\pi = \cos n\pi = (-1)^n \quad \text{and} \quad e^{-in\pi} = \cos n\pi - i \sin n\pi = \cos n\pi = (-1)^n \quad \because \sin n\pi = 0$$

$$= \frac{-(1-in\pi)}{2(1+n^2\pi^2)} \left\{ e^{-1} (-1)^n - e^1 (-1)^n \right\}$$

$$= \frac{(1 - in\pi)}{2(1 + n^2 \pi^2)} (-1)^n (e^1 - e^{-1})$$

$$C_n = \frac{(1 - in\pi)}{2(1 + n^2 \pi^2)} (-1)^n 2 \sinh(1) \quad \because \sinh \theta = \frac{e^\theta - e^{-\theta}}{2}$$

$$\therefore f(x) = \sum_{-\infty}^{\infty} \frac{(1 - in\pi)}{(1 + n^2 \pi^2)} (-1)^n \sinh(1) e^{in\pi x}$$

17. Calculate the first two harmonic of the Fourier series of $f(x)$ from the following data

x	0	30	60	90	120	150	180	210	240	270	300	330
f(x)	1.8	1.1	0.3	0.16	0.5	1.3	2.16	1.25	1.3	1.52	1.76	2.0

Solution:

x	0	30	60	90	120	150	180	210	240	270	300	330	360
f(x)	1.8	1.1	0.3	0.16	0.5	1.3	2.16	1.25	1.3	1.52	1.76	2.0	1.8

We know that $360 = 2\pi$

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{2\pi - 0}{2} = \pi \quad \therefore \ell = \pi$$

K=12

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \because \ell = \pi$$

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots$$

Where

$$a_0 = \frac{2}{K} \sum y \quad b_1 = \frac{2}{K} \sum y \sin x$$

$$a_1 = \frac{2}{K} \sum y \cos x \quad b_2 = \frac{2}{K} \sum y \sin 2x$$

$$a_2 = \frac{2}{K} \sum y \cos 2x$$

x	y	y cos x	y cos 2x	y sin x	y sin 2x
0	1.8	1.8	1.8	0	0
30	1.1	0.95	0.55	0.55	0.95
60	0.3	0.15	-0.15	0.26	0.26
90	0.16	0.00	-0.16	0.16	0.00
120	0.5	-0.25	-0.25	0.43	-0.43
150	1.3	-1.13	0.65	0.65	-1.12
180	2.16	-2.16	2.16	0.00	0.01
210	1.25	-1.08	0.62	-0.63	1.08
240	1.3	-0.65	-0.65	-1.13	1.12

270	1.52	0.00	-1.52	-1.52	-0.01
300	1.76	0.88	-0.87	-1.52	-1.53
330	2.0	1.73	1.01	-1.00	-1.73
Total	15.15	0.26	3.18	-3.74	-1.39

$$a_0 = \frac{2}{12}(15.15) = 2.52 \quad b_1 = \frac{2}{12}(-3.74) = -0.62$$

$$a_1 = \frac{2}{12}(0.26) = 0.043 \quad b_2 = \frac{2}{12}(-1.39) = -0.23$$

$$a_2 = \frac{2}{12}(3.18) = 0.53$$

$$f(x) = 1.26 + (0.043 \cos x - 0.62 \sin x) + (0.53 \cos 2x - 0.23 \sin 2x) + \dots$$

18. Find the first two harmonic of the Fourier series of f (x) given by

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
f (x)	1	1.4	1.9	1.7	1.5	1.2	1.0

Solution:

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
f (x)=y	1	1.4	1.9	1.7	1.5	1.2	1.0

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{2\pi - 0}{2} = \pi \quad \therefore \ell = \pi$$

$$K=6$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \because \ell = \pi$$

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots$$

Where

$$a_0 = \frac{2}{K} \sum y \quad b_1 = \frac{2}{K} \sum y \sin x$$

$$a_1 = \frac{2}{K} \sum y \cos x \quad b_2 = \frac{2}{K} \sum y \sin 2x$$

$$a_2 = \frac{2}{K} \sum y \cos 2x$$

x	y	y cos x	y cos 2x	y sin x	y sin 2x
0	1	1	1	0	0
$\frac{\pi}{3} = 60$	1.4	0.7	-0.7	1.212	1.212
$\frac{2\pi}{3} = 120$	1.9	-0.95	-0.95	1.65	-1.645
$\pi = 180$	1.7	-1.7	1.7	0	0
$\frac{4\pi}{3} = 240$	1.5	-0.75	-0.75	-1.299	1.299

$\frac{5\pi}{3} = 300$	1.2	0.6	-0.6	-1.039	-1.039
Total	8.7	-1.1	-0.3	0.5196	-0.1732

$$a_0 = \frac{2}{6}(8.7) = 2.9 \quad b_1 = \frac{2}{6}(0.5196) = 0.17$$

$$a_1 = \frac{2}{6}(-1.1) = -0.37 \quad b_2 = \frac{2}{6}(-0.1732) = -0.06$$

$$a_2 = \frac{2}{6}(-0.3) = -0.1$$

$$f(x) = 1.45 + (-0.37 \cos x + 0.17 \sin x) + (-0.1 \cos 2x - 0.06 \sin 2x) + \dots$$

19. Find the first three harmonic of the Fourier series of f (x) given by

X	0	1	2	3	4	5
f (x)	9	18	24	28	26	20

Solution:

X	0	1	2	3	4	5	6
f (x)	9	18	24	28	26	20	9

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{6 - 0}{2} = 3 \quad \therefore \ell = 3$$

$$K=6$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3}$$

$$f(x) = \frac{a_0}{2} + \left(a_1 \cos \frac{\pi x}{3} + b_1 \sin \frac{\pi x}{3} \right) + \left(a_2 \cos \frac{2\pi x}{3} + b_2 \sin \frac{2\pi x}{3} \right) + \left(a_3 \cos \frac{3\pi x}{3} + b_3 \sin \frac{3\pi x}{3} \right) + \dots$$

Where

$a_0 = \frac{2}{K} \sum y$ $a_1 = \frac{2}{K} \sum y \cos \left(\frac{\pi x}{3} \right)$ $a_2 = \frac{2}{K} \sum y \cos \left(\frac{2\pi x}{3} \right)$ $a_3 = \frac{2}{K} \sum y \cos \left(\frac{3\pi x}{3} \right)$				$b_1 = \frac{2}{K} \sum y \sin \left(\frac{\pi x}{3} \right)$ $b_2 = \frac{2}{K} \sum y \sin \left(\frac{2\pi x}{3} \right)$ $b_3 = \frac{2}{K} \sum y \sin \left(\frac{3\pi x}{3} \right)$			
x	y	$y \cos \frac{\pi x}{3}$ (or) $y \cos 60x$	$y \cos \frac{2\pi x}{3}$ $y \cos 120x$	$y \cos \frac{3\pi x}{3}$ $y \cos 180x$	$y \sin \frac{\pi x}{3}$ $y \sin 60x$	$y \sin \frac{2\pi x}{3}$ $y \sin 120x$	$y \sin \frac{3\pi x}{3}$ $y \cos 180x$
0	9	9	9	9	0	0	0
1	18	9	-9	-18	15.7	15.6	0
2	24	-12	-24	24	20.9	-20.784	0
3	28	-28	28	-28	0	0	0
4	26	-13	-13	26	-22.6	22.6	0

5	20	10	-10	-20	-17.4	-17.4	0
Total	125	-25	-19	-7	-3.4	20.8	0

$$a_0 = \frac{2}{6}(125) = 41.66 \quad b_1 = \frac{2}{6}(-3.4) = -1.13$$

$$a_1 = \frac{2}{6}(-25) = -8.33 \quad b_2 = \frac{2}{6}(20.8) = 6.9$$

$$a_2 = \frac{2}{6}(-19) = -6.33 \quad b_3 = \frac{2}{6}(0) = 0$$

$$a_3 = \frac{2}{6}(-7) = -2.3$$

The required Harmonic of the Fourier series be

$$f(x) = \frac{41.66}{2} + \left(-8.33 \cos \frac{\pi x}{3} + -1.13 \sin \frac{\pi x}{3} \right) + \left(-6.33 \cos \frac{2\pi x}{3} + 6.9 \sin \frac{2\pi x}{3} \right) + \left(-2.3 \cos \frac{3\pi x}{3} + 0 \right) + \dots$$

$$f(x) = 20.83 + \left(-8.33 \cos \frac{\pi x}{3} + -1.13 \sin \frac{\pi x}{3} \right) + \left(-6.33 \cos \frac{2\pi x}{3} + 6.9 \sin \frac{2\pi x}{3} \right) + \left(-2.3 \cos \frac{3\pi x}{3} \right) + \dots$$

20. The following table gives the variations of a periodic function over a period T

x	0	T/6	T/3	T/2	2T/3	5T/6	T
f(x)	1.98	1.3	1.05	1.3	-0.88	-0.25	1.98

find f(x) upto first harmonic.

Solution:

Assume $X = \frac{2\pi x}{T}$

X	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{2\pi}{3}$
y=f(θ)	1.98	1.3	1.05	1.3	-0.88	-0.25	1.98

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{2\pi - 0}{2} = \pi \quad \therefore \ell = \pi$$

$$K=6$$

$$f(X) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi X}{\ell} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi X}{\ell}$$

$$f(X) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nX + \sum_{n=1}^{\infty} b_n \sin nX$$

$$f(x) = \frac{a_0}{2} + (a_1 \cos nX + b_1 \sin nX) + (a_2 \cos 2X + b_2 \sin 2X) + (a_3 \cos 3X + b_3 \sin 3X) + \dots$$

Where

$a_0 = \frac{2}{K} \sum y \quad a_1 = \frac{2}{K} \sum y \cos X$				$b_1 = \frac{2}{K} \sum y \sin X$			
x	y	y cos X	y sin X				
0	1.98	1.98	0				
$\frac{\pi}{3}$	1.30	0.65	1.1258				

$\frac{2\pi}{3}$	1.05	-0.525	0.9093
π	1.30	-1.3	0
$\frac{4\pi}{3}$	-0.88	0.44	0.762
$\frac{5\pi}{3}$	-0.25	-0.125	0.2165
Total	4.6	1.12	3.013

$$a_0 = \frac{2}{6} \sum y = \frac{4.6}{3} = 1.5 \quad a_1 = \frac{2}{6} (1.12) = 0.37 \quad b_1 = \frac{2}{6} (3.013) = 1.005$$

$$\therefore \boxed{f(x) = 0.75 + 0.37 \cos X + 1.005 \sin X}$$