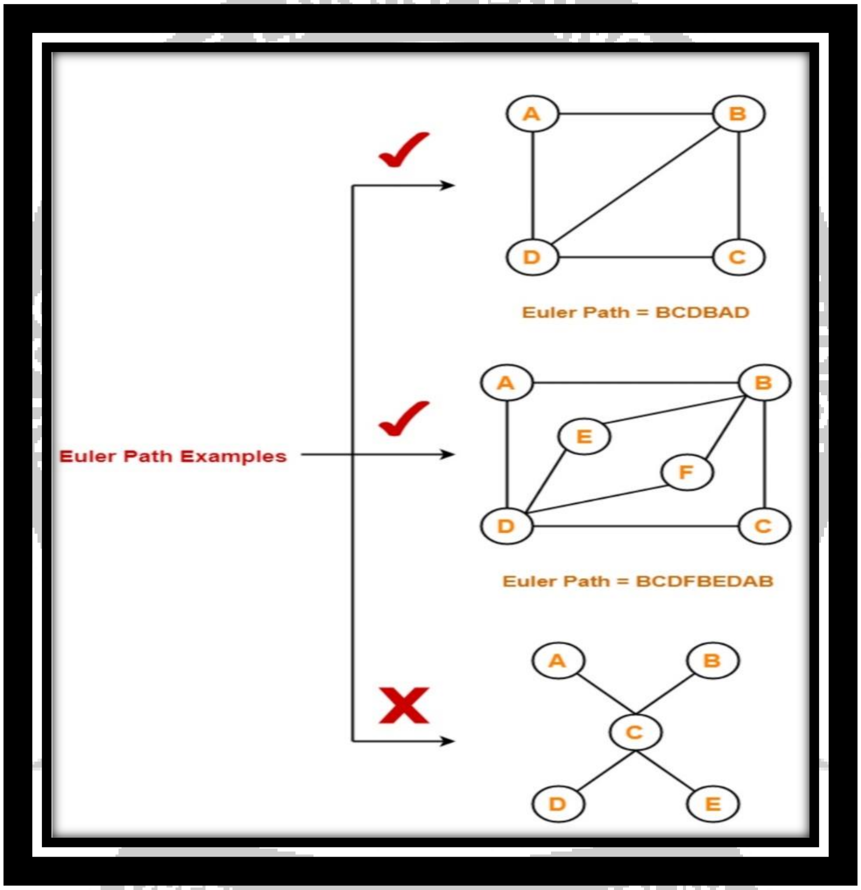
Euler graph and Hamilton graph:

Euler path:

A path of a graph G is called an Eulerian path, if it contains each edge of the graph exactly once.



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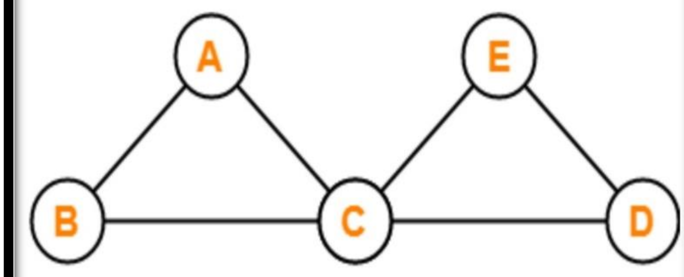


Example of Euler Graph

Eulerian Circuit or Eulerian Cycle:

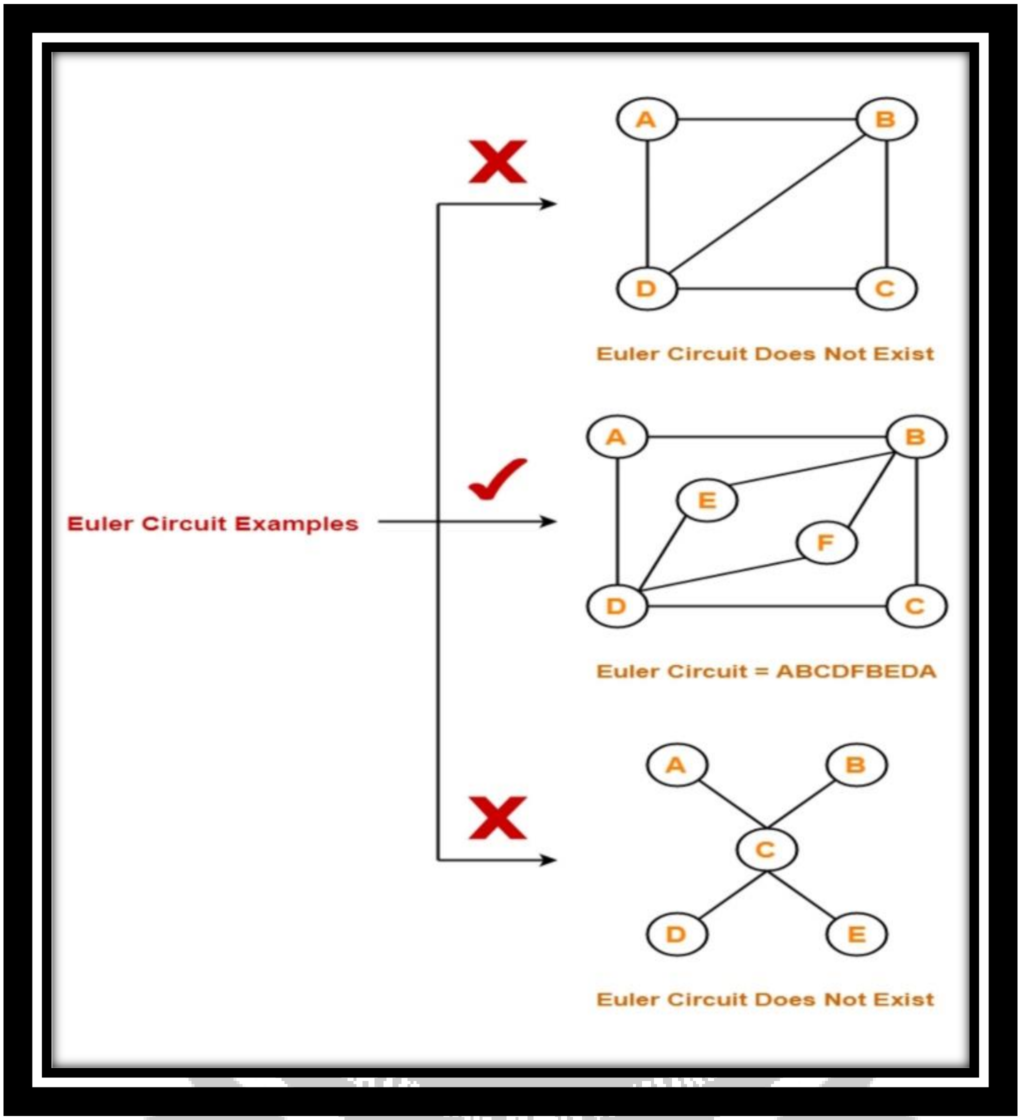
A circuit or cycle of a graph G is called Eulerian circuit or cycle, if it includes each edge of G exactly once.

Here starting and ending vertex are same.

An Eulerian circuit or cycle should satisfies the following conditions:

Starting and ending points (vertices) are same.

Cycle should contain all the edges of graph but exactly once.



Eulerian Graph or Euler graph:

Any graph containing an Eulerian circuit or cycle is called an Eulerian graph.

Theorem:1

A connected graph is Euler graph (contains Eulerian circuit) if and only if each of its vertices is of even degree.

Proof:

Let G be any graph having an Eulerian circuit and let “C” be an Eulerian circuit of G with origin vertex as u. Each time a vertex occurs as an internal vertex of C, then two of the edges incident with v are accounted for degree.

We, get, for internal vertex $v \in v(G)$

$d(v) = 2 \times$ number of times v occur inside the Euler circuit C

= even degree

And, since an Euler circuit C contains every edge of G and C starts and ends at u.

$d(u) = 2 + 2 + \times$ number of times u occur inside C.

= even degree Hence G has all the vertices of even degree.

Conversely, assume each of its vertices has an even degree.

Claim:

G has an Eulerian circuit.

Assume G be a connected graph which is not having an Euler circuit, with all vertices of even degree and less number of edges. That is, any graph having less number of edges than G, then it has an Eulerian circuit. Since each vertex of G has atleast two, therefore G contains closed path. Let C be a closed path of maximum possible length in G. If C itself has all the edges of G, then G itself an Euler circuit in G.

By assumption, C is not an Euler circuit of G and $G - E(C)$ has some component G' with $|E(G')| > 0$. C has less number of edges than G, therefore C itself is an Eulerian, and C has all the vertices of even degree.

Since $|E(G')| < |E(G)|$, therefore G' has an Euler circuit C'. Because G is connected, there is a vertex v in both C and C'. Now join C and C' and traverse all the edges of C and C' with common vertex v, we get CC' is a closed path in G and $E(CC') > E(C)$ which is not possible choices of C.

Hence G has an Eulerian circuit.

Hence G is a Euler graph.

Hence the proof.

Theorem:2

Prove that if a graph G has not more than two vertices of odd degree, then there can be Euler path in G.

Proof:

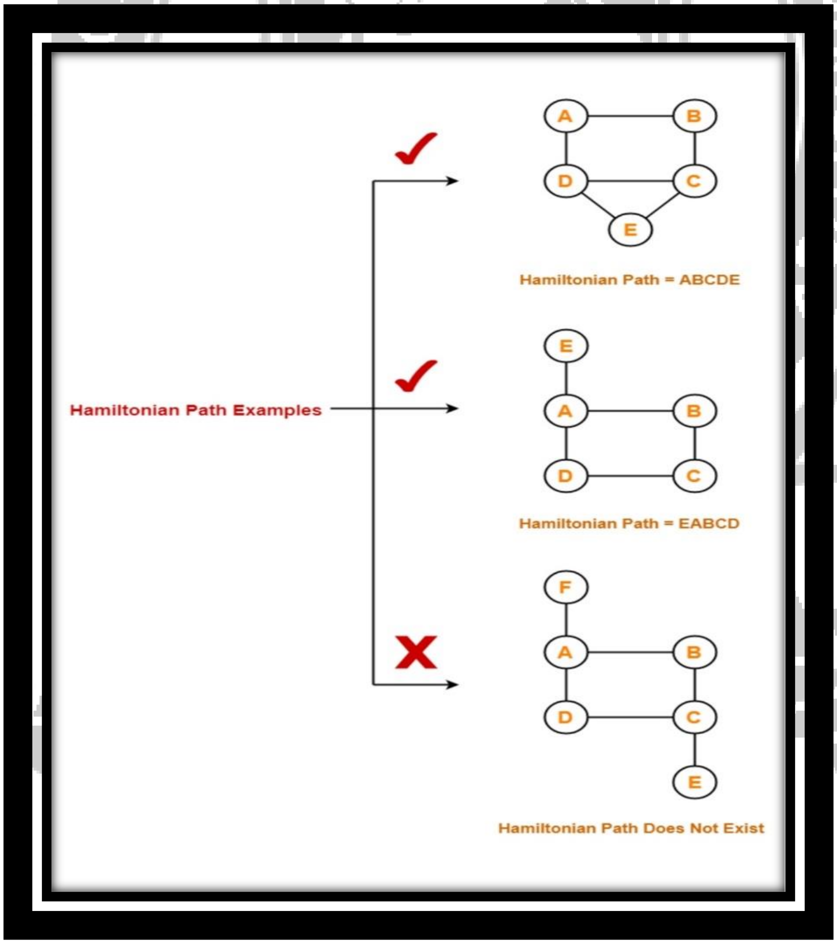
Let the odd degree vertices be labelled as V and W in any arbitrary order. Add an edge of G between the vertex pair (V, W) to form a new graph G'.

Now every vertex of G' is of even degree and hence G' has an Eulerian trial T. If the edge that we added to G is now removed from T, it will split into an open trail containing all edges of G which is nothing but an Euler path in G.

Hamiltonian Graph:

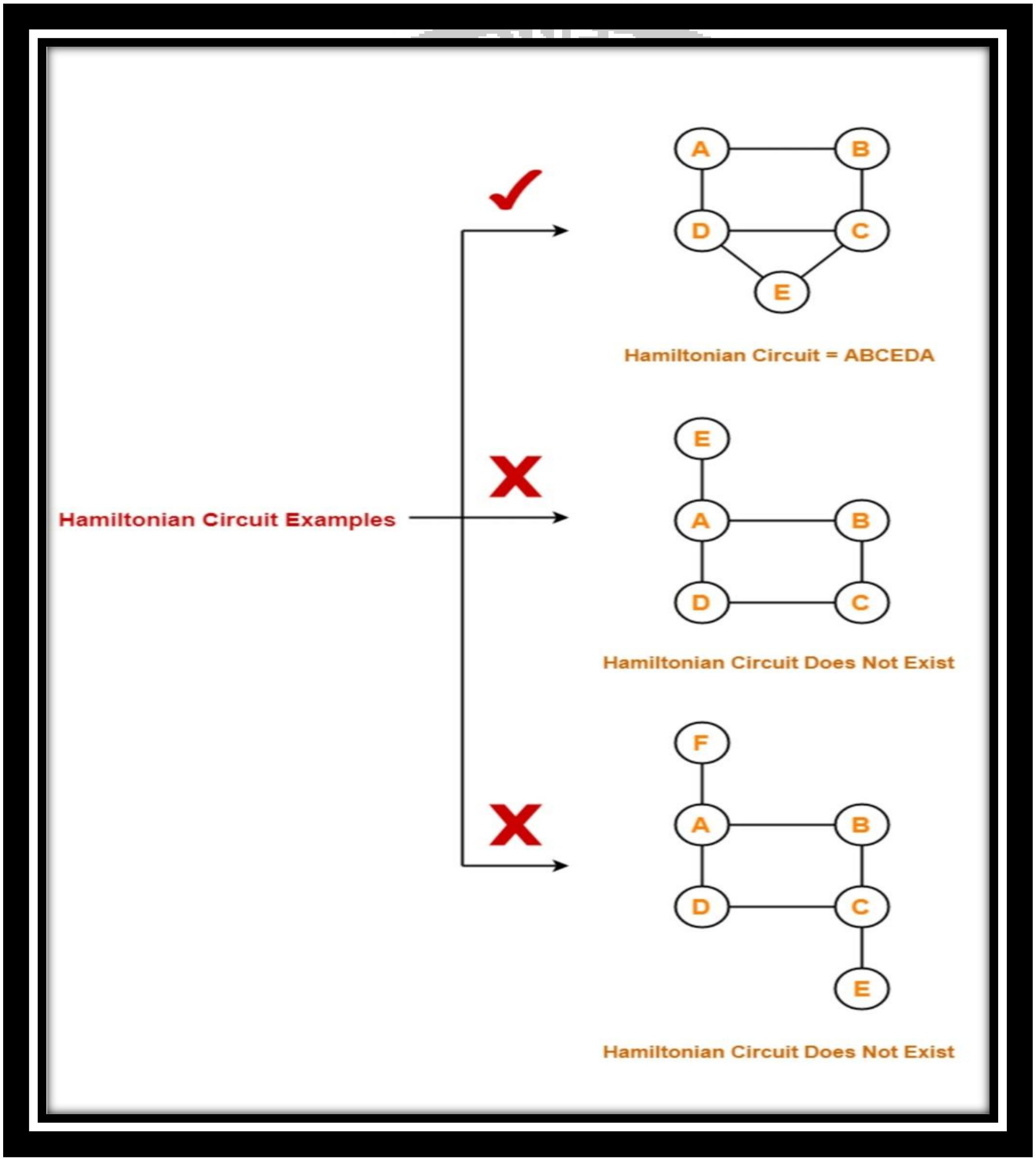
Hamiltonian Path:

A path of a graph G is called a Hamiltonian path, if it includes each vertex of G exactly once.



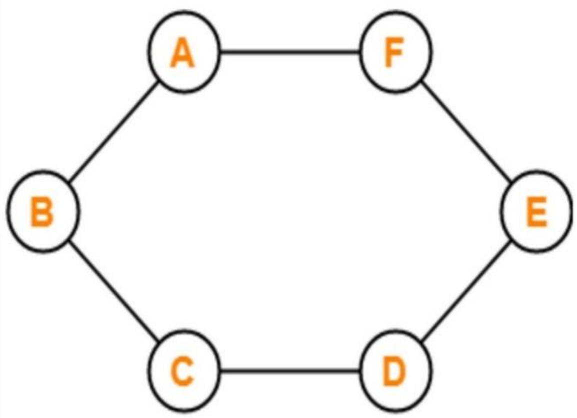
Hamiltonian Circuit or Cycle:

A circuit of a graph G is called a Hamiltonian circuit, if it includes each vertex of G exactly once, except the starting and ending vertices.



Hamiltonian graph:

Any graph containing a Hamiltonian circuit or cycle is called a Hamiltonian graph.



Example of Hamiltonian Graph

Properties:

(i) A Hamiltonian circuit contains a Hamiltonian path, but a graph containing a Hamiltonian path need not have a Hamiltonian cycle.

(ii) By deleting any one edge from Hamiltonian cycle, we can get Hamiltonian path.

(iii) A graph may contain more than one Hamiltonian cycle.

(iv) A complete graph k_n, will always have a Hamiltonian cycle, when $n \geq 3$.

(v) A graph with a vertex of degree one cannot have a Hamiltonian cycle.

Theorem: 1

Let G be a simple indirected graph with n vertices. Let u and v be two nonadjacent vertices in G such that $deg(u) + deg(v) \geq n$ in G. Show that G is Hamiltonian if and only if $G + uv$ is Hamiltonian.

Proof:

If G is Hamiltonian, then obviously $G + uv$ is Hamiltonian.

Conversely, suppose that $G + uv$ is Hamiltonian, but G is not.

Then by Dirac theorem, we have $deg(u) + deg(v) < n$

Which is a contradiction to our assumption.

Thus $G + uv$ is Hamiltonian implies G is Hamiltonian.

Hence the proof.