2.1 Mathematical induction:

Statement of the principle of Mathematical Induction

Let $P(n)$be statement involving the natural number “n”.

If P(1) is true.

Under the assumption that when P(k) is true, P(k+1) is true, then we conclude that a statement P(n) is true for all natural number “n”.

Steps to prove that a statement P(n) is true for all natural numbers

Step:1 We must prove that P(1) is true.

Step:2 By assuming P(k) is true, we must prove that P(k+1) is also true.

NOTE:

Step:1 is known as the basic step.

Step:2 is known as inductive step.

Problems on Mathematical Induction:

1. Show that $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$ using mathematical induction.

Solution:

Let S be the set of positive integers.

To prove $p(1)$ is true.

When $n = 1$

RHS $\Rightarrow \frac{n(n+1)}{2} = \frac{1(1+1)}{2} = 1 = \text{LHS}$

Hence $p(1)$ is true.

Assume that $p(k)$ is true.

$$1 + 2 + \ldots + k = \frac{k(k+1)}{2} \quad \ldots (1)$$

To prove $p(k+1)$ is true.

Adding $k + 1$ on both sides

$$\Rightarrow 1 + 2 + \ldots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

Hence $p(k+1)$ is true.

2. Show that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Solution:

Let S be the set of positive integers.

To prove $p(1)$ is true.

When $n = 1$

RHS $\Rightarrow \frac{n(n+1)(2n+1)}{6} = \frac{1(1+1)(2+1)}{6} = 1 =$ LHS

Hence $p(1)$ is true.

Assume that $p(k)$ is true.

$$1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6} \quad \dots (1)$$

To prove $p(k+1)$ is true.

$$1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

Adding $(k+1)^2$ on both sides

$$\Rightarrow 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right]$$

$$= \frac{(k+1)}{6}[k(2k+1) + 6(k+1)]$$

$$= \frac{(k+1)}{6}[2k^2 + k + 6k + 6]$$

$$= \frac{(k+1)}{6}[2k^2 + 7k + 6]$$

$$= \frac{(k+1)}{6}[2k^2 + 4k + 3k + 6]$$

$$= \frac{(k+1)}{6}[2k(k + 2) + 3(k + 2)]$$

$$= \frac{(k+1)}{6}[(k + 2) + (2k + 3)]$$

Hence $p(k + 1)$ is true.

3. Show that $\mathbf{1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}}$

Solution:

Let S be the set of positive integers.

To prove $p(1)$ is true.

When $n = 1$

RHS $\Rightarrow \frac{n^2(n+1)^2}{4} = \frac{1^2(1+1)^2}{4} = 1 =$ LHS

Hence $p(1)$ is true.

Assume that $p(k)$ is true.

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2(k+1)^2}{4} \qquad \ldots (1)$$

To prove $p(k+1)$ is true.

$$1^3+2^3+3^3+\ldots+k^3+(k+1)^3=\frac{(k+1)^2(k+2)^2}{4}$$

Adding $(k+1)^3$ on both sides

$$\Rightarrow 1^3+2^3+3^3+\ldots+k^3+(k+1)^3=\frac{k^2(k+1)^2}{4}+(k+1)^3$$

$$=\frac{k^2(k+1)^2+4(k+1)^3}{4}$$

$$=\frac{(k+1)^2}{4}[k^2+4(k+1)]$$

$$=\frac{(k+1)^2}{4}[k^2+4k+4]$$

$$=\frac{(k+1)^2}{4}[k^2+2k+2k+4]$$

$$=\frac{(k+1)^2}{4}[k(k+2)+2(k+2)]$$

$$=\frac{(k+1)^2}{4}[(k+2)+(k+2)]$$

$$=\frac{(k+1)^2(k+2)^2}{4}$$

Hence $p(k+1)$ is true.

4.Prove that n^3-n is divisible by 3, using mathematical induction .

Solution:

Let S be the set of positive integers.

To prove $p(1)$ is true.

When $n = 1$

RHS $\Rightarrow n^3 - n = 1^3 - 1 = 0$ is divisible by 3.

Hence $p(1)$ is true.

Assume that $p(k)$ is true.

$k^3 - k$ is divisible by 3.

$\Rightarrow k^3 - k = 3m$

$\Rightarrow k^3 = 3m + k \; \ldots (1)$

To prove $p(k+1)$ is true.

$(k+1)^3 - (k+1)$ is divisible by 3.

$$\Rightarrow k^3 + 1 + 3k^2 + 3k - k - 1$$

$$\Rightarrow k^3 + 3k^2 + 2k$$

$$\Rightarrow (3m + k) + 3k^2 + 2k$$

$$\Rightarrow 3m + 3k^2 + 3k$$

$\Rightarrow 3(m + k^2 + k)$ is divisible by 3.

Hence $p(k+1)$ is true.

5. Prove that $8^n - 3^n$ is a multiple of 5. .

Solution:

Let S be the set of positive integers.

To prove $p(1)$ is true.

When $n = 1$

RHS $\Rightarrow 8^n - 3^n = 8^1 - 3^1 = 5$ is a multiple of 5 which is true.

Hence $p(1)$ is true.

Assume that $p(k)$ is true.

$8^k - 3^k$ is a multiple of 5.

$\Rightarrow 8^k - 3^k = 5m$

$\Rightarrow 8^k = 5m + 3^k$ . . . (1)

To prove $p(k+1)$ is true.

$8^{k+1} - 3^{k+1}$ is a multiple of 5.

$$\Rightarrow 8 \cdot 8^k - 3 \cdot 3^k$$

$$\Rightarrow (5m + 3^k) \cdot 8 - 3 \cdot 3^k$$

$$\Rightarrow 5 \cdot 8m + 8 \cdot 3^k - 3 \cdot 3k$$

$$\Rightarrow 5 \cdot 8m + 5 \cdot 3^k$$

$\Rightarrow 5(8m + 3^k)$ is a multiple of 5.

Hence $p(k+1)$ is true.

6. State and prove Handshaking theorem.

Suppose there are "n" people in a room, $n \geq 1$ and that they all shake hands with one another, prove that $\frac{n(n-1)}{2}$ handshakes will have accured.

Solution:

Let S be the set of positive integers.

To prove $p(1)$ is true.

When $n = 1$

$p(1) = \frac{n(n-1)}{2} = \frac{1(1-1)}{2} = 0$

$\Rightarrow$ there is no handshake accured which means there is only one person.

Hence $p(1)$ is true.

Assume that $p(k)$ is true.

$p(k) = \frac{k(k-1)}{2} \quad \ldots (1)$

To prove $p(k+1)$ is true.

$$p(k+1) = \frac{(k+1)k}{2}$$

Suppose if one person entered into the room then he will shake his hand with "k" other person whenever $p(k)$ is true.

Hence $p(k+1)$ is true by mathematical induction.

The Well – Ordering Property:

The validity of mathematical induction follows from the following fundamental axioms about the set of integers.

Every non – empty set of non – negative integers has a least element.

The well ordering property can often be used directly in the proof.