4.2. Quantum Entropy

Introduction

Entropy measures uncertainty or mixedness. In classical information theory, Shannon entropy measures the average information associated with a random variable. Quantum information theory uses Von Neumann entropy to describe the uncertainty of a quantum state and to study information in quantum systems.

Definition

Quantum entropy is a mathematical measure of the uncertainty or mixedness of a quantum state. The principal measure is Von Neumann entropy, $S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho)$.

4.2.1 Density matrix

A density matrix ρ describes a quantum state when the system is pure, mixed, or part of a larger system. It is Hermitian, positive semidefinite and has trace one. A pure state has $\rho = |\psi\rangle\langle\psi|$. A mixture has $\rho = \Sigma_i p_i|\psi_i\rangle\langle\psi_i|$, where p_i are probabilities that sum to one.

4.2.2 Von Neumann entropy

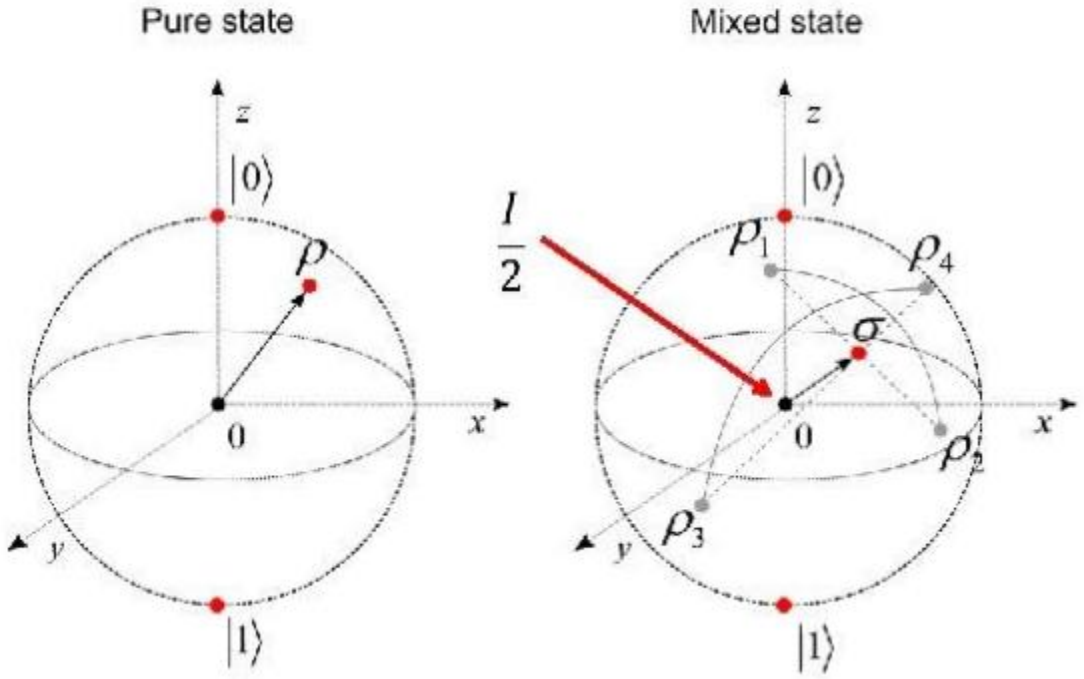
The formula $S(\rho) = -\mathrm{Tr}(\rho \log_2\rho)$ can also be written as $S(\rho) = -\Sigma_i \lambda_i \log_2\lambda_i$, where λ_i are the eigenvalues of ρ. A zero eigenvalue contributes zero by convention. Entropy is measured in bits when the logarithm is base two.

4.2.3 Pure-state and maximally mixed examples

A pure qubit such as $|0\rangle$ has eigenvalues 1 and 0, so its entropy is zero. A maximally mixed qubit $\rho = I/2$ has eigenvalues 1/2 and 1/2, so $S(\rho) = -2(1/2 \log_2(1/2)) = 1$ bit. Thus a qubit's entropy lies between zero and one bit.

Pure and mixed states

- The density matrix is not unique!



- Pure state: The quantum system is completely described by one state vector. Its Von Neumann entropy is zero.
- Mixed state: The system is described by a statistical mixture of possible states. Its entropy can be greater than zero.
- Maximum mixed state: A qubit with equal probabilities of 0 and 1 has entropy of one bit.

4.2.4 Important properties

Entropy is non-negative; it is zero for a pure state; it is invariant under unitary transformations; and for a system of dimension d it is at most $\log_2 d$. For

independent systems entropy is additive. For a joint state, subadditivity states $S(AB) \leq S(A) + S(B)$. Entanglement can produce correlations that make the joint entropy behave differently from the sum of local entropies.

Properties of Quantum Entropy

1. **Non-negative:** Quantum entropy is always greater than or equal to zero.
2. **Pure State:** A pure quantum state has zero Von Neumann entropy.
3. **Maximum Entropy:** A completely mixed qubit has entropy of one bit.
4. **Additivity:** For independent quantum systems, the entropy of their combined state is the sum of their individual entropies.
5. **Unitary Invariance:** Quantum entropy does not change under unitary operations.
6. **Concavity:** Mixing quantum states cannot decrease entropy below the average entropy of the individual states.
7. **Subadditivity:** The entropy of a combined system is not greater than the sum of the individual entropies.

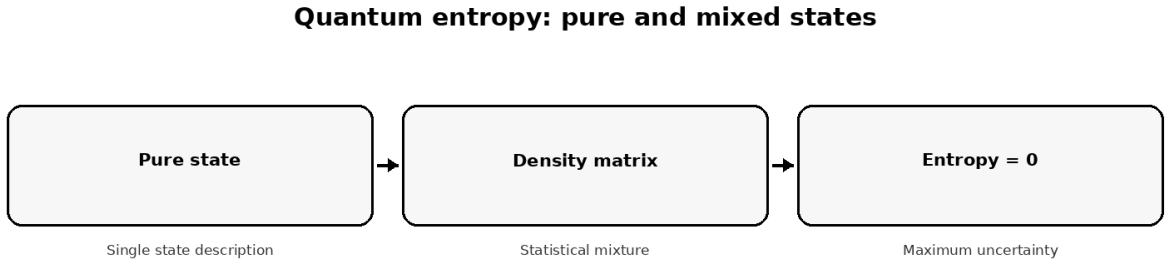
4.2.5 Other quantum entropy measures

Quantum relative entropy compares states: $D(\rho||\sigma) = Tr[\rho(\log \rho - \log \sigma)]$ when the support conditions are satisfied. Conditional entropy is $S(A|B) = S(AB) - S(B)$. Rényi entropies form a family of measures parameterised by an order. These measures are useful for different tasks, including distinguishability, correlations and security analysis.

4.2.6 Classical and quantum entropy

Shannon entropy is $H(X) = -\Sigma_x p(x)\log_2 p(x)$. Von Neumann entropy is its quantum generalisation for density operators. For a density matrix that is diagonal in a basis, Von Neumann entropy equals the Shannon entropy of its eigenvalue distribution.

Diagram



The diagram contrasts a pure state, which is fully specified and has zero Von Neumann entropy, with a mixed state, which represents a statistical mixture. The entropy value depends on the eigenvalues of the density matrix.

Advantages

- Quantifies uncertainty and mixedness in quantum states.
- Helps analyse entanglement, communication limits and information leakage.
- Provides the theoretical basis for quantum source compression.
- Connects quantum information with thermodynamics.

Disadvantages / limitations

- Exact calculation can be demanding for large density matrices.
- Entropy alone does not describe every feature of a quantum state.
- Experimental estimation often requires many measurements.
- Different entropy measures answer different questions.

Applications

- Determining quantum source compression rates.
- Studying entanglement and correlations.
- Evaluating information leakage in quantum cryptography.

- Analysing quantum communication and channel capacities.
- Quantum thermodynamics and statistical physics.

Thus Quantum entropy quantifies uncertainty and mixedness. It is essential for understanding quantum compression, entanglement, communication limits and quantum thermodynamics.