

### 5.3 Triple Integrals- Volume of Solids

Triple integration in cartesian co-ordinates is defined over a region R is defined by

$$\iiint_R f(x, y, z) dx dy dz \text{ or } \iiint_R f(x, y, z) dV \text{ or } \iiint_R f(x, y, z) d(x, y, z).$$

#### Type I – Problems on Triple Integrals

**Example:**

Evaluate  $\int_0^a \int_0^b \int_0^c (x + y + z) dz dy dx$

**Solution:**

$$\begin{aligned} \int_0^a \int_0^b \int_0^c (x + y + z) dz dy dx &= \int_0^a \int_0^b \left[ xz + yz + \frac{z^2}{2} \right]_0^c dy dx \\ &= \int_0^a \int_0^b \left( xc + yc + \frac{c^2}{2} \right) dy dx \\ &= \int_0^a \left[ xcy + \frac{y^2c}{2} + \frac{c^2y}{2} \right]_0^b dx \\ &= \int_0^a \left( xbc + \frac{b^2c}{2} + \frac{bc^2}{2} \right) dx \\ &= \left[ \frac{x^2bc}{2} + \frac{b^2cx}{2} + \frac{bc^2x}{2} \right]_0^a \\ &= \left[ \frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2} \right] \\ &= \frac{abc}{2} (a + b + c) \end{aligned}$$

**Example:**

Evaluate  $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x dz dx dy$

**Solution:**

$$\begin{aligned} \int_0^1 \int_{y^2}^1 \int_0^{1-x} x dz dx dy &= \int_0^1 \int_{y^2}^1 [xz]_0^{1-x} dx dy \\ &= \int_0^1 \int_{y^2}^1 x(1-x) dx dy \\ &= \int_0^1 \int_{y^2}^1 (x - x^2) dx dy \\ &= \int_0^1 \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_{y^2}^1 dy \end{aligned}$$

$$\begin{aligned}
 &= \int_0^1 \left( \frac{1}{2} - \frac{1}{3} - \frac{y^4}{2} + \frac{y^6}{3} \right) dy \\
 &= \left[ \frac{y}{2} - \frac{y}{3} - \frac{y^5}{10} + \frac{y^7}{21} \right]_0^1 \\
 &= \frac{1}{2} - \frac{1}{3} - \frac{1}{10} + \frac{1}{21} \\
 &= \frac{105 - 70 - 21 + 10}{210} = \frac{24}{210} = \frac{4}{35}
 \end{aligned}$$

**Example:**

Evaluate  $\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z dx dy dz$

**Solution:**

$$\begin{aligned}
 \int_0^1 \int_0^{1-x} \int_0^{x+y} e^z dx dy dz &= \int_0^1 \int_0^{1-x} \int_0^{x+y} e^z dz dy dx \\
 &= \int_0^1 \int_0^{1-x} [e^z]_0^{x+y} dy dx \\
 &= \int_0^1 \int_0^{1-x} (e^{x+y} - 1) dy dx \\
 &= \int_0^1 [e^{x+y} - y]_0^{1-x} dx \\
 &= \int_0^1 (e^{x+1-x} - 1 + x - e^x) dx \\
 &= \int_0^1 (e - 1 + x - e^x) dx \\
 &= \left[ ex - x + \frac{x^2}{2} - e^x \right]_0^1 \\
 &= e - 1 + \frac{1}{2} - e - 0 + 0 - 0 + 1 \\
 &= \frac{1}{2}
 \end{aligned}$$

**Example:**

Evaluate  $\int_0^a \int_0^{\sqrt{a^2-y^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz dx dy$

**Solution:**

$$\begin{aligned}
 \int_0^a \int_0^{\sqrt{a^2-y^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz dx dy &= \int_0^a \int_0^{\sqrt{a^2-y^2}} [z]_0^{\sqrt{a^2-x^2-y^2}} dx dy \\
 &= \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2} dx dy
 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{(\sqrt{a^2-y^2})^2 - x^2} dx dy \\
 &= \int_0^a \left[ \frac{x}{2} \sqrt{(a^2-y^2) - x^2} + \frac{a^2-y^2}{2} \sin^{-1} \frac{x}{\sqrt{a^2-y^2}} \right]_0^{\sqrt{a^2-y^2}} dy \\
 &= \int_0^a \left( 0 + \frac{a^2-y^2}{2} \sin^{-1} 1 - 0 - 0 \right) dy \\
 &= \int_0^a \left( \frac{a^2-y^2}{2} \right) \frac{\pi}{2} dy \\
 &= \frac{\pi}{4} \int_0^a (a^2 - y^2) dy \\
 &= \frac{\pi}{4} \left[ a^2 y - \frac{y^3}{3} \right]_0^a \\
 &= \frac{\pi}{4} \left[ a^3 - \frac{a^3}{3} - 0 \right] \\
 &= \frac{\pi}{4} \left( \frac{2a^3}{3} \right) \\
 &= \frac{\pi a^3}{6}
 \end{aligned}$$

**Example:**

**Evaluate**  $\int_0^{\log a} \int_0^x \int_0^{x+y} e^{x+y+z} dz dy dx$

**Solution:**

$$\begin{aligned}
 \int_0^{\log a} \int_0^x \int_0^{x+y} e^{x+y+z} dz dy dx &= \int_0^{\log a} \int_0^x [e^{x+y+z}]_0^{x+y} dy dx \\
 &= \int_0^{\log a} \int_0^x (e^{2(x+y)} - e^{x+y}) dy dx \\
 &= \int_0^{\log a} \left[ \frac{e^{2(x+y)}}{2} - e^{x+y} \right]_0^x dx \\
 &= \int_0^{\log a} \left( \frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x \right) dx \\
 &= \int_0^{\log a} \left( \frac{e^{4x}}{2} - \frac{3}{2} \times \frac{e^{2x}}{2} + e^x \right) dx \\
 &= \left[ \frac{e^{4x}}{8} - \frac{3e^{2x}}{8} + e^x \right]_0^{\log a} \\
 &= \frac{e^{4 \log a}}{8} - \frac{3}{8} e^{2 \log a} + e^{\log a} - \frac{1}{8} + \frac{3}{8} - 1
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{e^{\log a^4}}{8} - \frac{3}{8} e^{\log a^2} + a + \left(\frac{-1+6-8}{8}\right) \\
 &= \frac{a^4}{8} - \frac{3a^2}{8} + a - \frac{6}{8} \quad [\because e^{\log x} = x]
 \end{aligned}$$

**Type:II Problem on Triple Integral if region is given**

**Example:**

**Express the region  $x \geq 0, y \geq 0, z \geq 0, x^2 + y^2 + z^2 \leq 1$  by triple integration.**

**Solution:**

For the given region,  $z$  varies from 0 to  $\sqrt{1 - x^2 - y^2}$

$y$  varies from 0 to  $\sqrt{1 - x^2}$

$x$  varies from 0 to 1

$$\therefore I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dz dy dx$$

**Example:**

**Evaluate  $\iiint x^2 y z dx dy dz$  taken over the tetrahedron bounded by the planes**

**$x = 0, y = 0, z = 0$  and  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ .**

**Solution:**

Given  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ .

Limits are ,  $z$  varies from 0 to  $c \left(1 - \frac{x}{a} - \frac{y}{b}\right)$

$y$  varies from 0 to  $b \left(1 - \frac{x}{a}\right)$

$x$  varies from 0 to  $a$

$$\begin{aligned}
 \iiint x^2 y z dx dy dz &= \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} x^2 dz dy dx \\
 &= \int_0^a \int_0^{b(1-\frac{x}{a})} \left[ x^2 y \frac{z^2}{2} \right]_0^{c(1-\frac{x}{a}-\frac{y}{b})} dy dx \\
 &= \int_0^a \int_0^{b(1-\frac{x}{a})} \left( \frac{x^2 y c^2 \left(1 - \frac{x}{a} - \frac{y}{b}\right)^2}{2} \right) dy dx
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{c^2}{2} \int_0^a \int_0^{bk} x^2 y \left(k - \frac{y}{b}\right)^2 dy dx && \left[ \because k = 1 - \frac{x}{a} \right] \\
 &= \frac{c^2}{2} \int_0^a \int_0^{bk} x^2 y \left(k^2 + \frac{y^2}{b^2} - \frac{2ky}{b}\right) dy dx \\
 &= \frac{c^2}{2} \int_0^a \int_0^{bk} x^2 \left(yk^2 + \frac{y^3}{b^2} - \frac{2ky^2}{b}\right) dy dx \\
 &= \frac{c^2}{2} \int_0^a x^2 \left[\frac{k^2 y^2}{2} + \frac{y^4}{4b^2} - \frac{2ky^3}{3b}\right]_0^{bk} dx \\
 &= \frac{c^2}{2} \int_0^a x^2 \left(\frac{b^2 k^4}{2} + \frac{b^4 k^4}{4b^2} - \frac{2b^3 k^4}{3b}\right) dx \\
 &= \frac{c^2}{2} \int_0^a x^2 \left(\frac{b^2 k^4}{2} + \frac{b^2 k^4}{4} - \frac{2b^2 k^4}{3}\right) dx \\
 &= \frac{b^2 c^2}{2} \int_0^a k^4 x^2 \left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right) dx \\
 &= \frac{b^2 c^2}{24} \int_0^a x^2 \left(1 - \frac{x}{a}\right)^4 dx \\
 &\quad \left[ \because (1-x)^n = 1 - nx + \frac{n(n-1)}{2!} x^2 - \frac{n(n-1)(n-2)}{3!} x^3 + \dots \right] \\
 &= \frac{b^2 c^2}{24} \int_0^a x^2 \left(1 - \frac{4x}{a} + \frac{4 \times 3}{2!} \times \frac{x^2}{a^2} - \frac{4 \times 3 \times 2}{3!} \times \frac{x^3}{a^3} + \frac{4 \times 3 \times 2 \times 1}{4!} \times \frac{x^4}{a^4}\right) dx \\
 &= \frac{b^2 c^2}{24} \int_0^a \left(x^2 - \frac{4x^3}{a} + \frac{6x^4}{a^2} - \frac{4x^5}{a^3} + \frac{x^6}{a^4}\right) dx \\
 &= \frac{b^2 c^2}{24} \left[\frac{x^3}{3} - \frac{4x^4}{4a} + \frac{6x^5}{5a^2} - \frac{4x^6}{6a^3} + \frac{x^7}{7a^4}\right]_0^a \\
 &= \frac{b^2 c^2}{24} \left[\frac{a^3}{3} - \frac{a^4}{a} + \frac{6a^5}{5a^2} - \frac{2a^6}{3a^3} + \frac{a^7}{7a^4}\right] \\
 &= \frac{b^2 c^2}{24} \left[\frac{a^3}{3} - a^3 + \frac{6a^3}{5} - \frac{2a^3}{3} + \frac{a^3}{7}\right] \\
 &= \frac{a^3 b^2 c^2}{24} \left[\frac{1}{3} - 1 + \frac{6}{5} - \frac{2}{3} + \frac{1}{7}\right] \\
 &= \frac{a^3 b^2 c^2}{24} \left(\frac{35 - 105 + 126 - 70 + 15}{105}\right) \\
 &= \frac{a^3 b^2 c^2}{24} \left(\frac{1}{105}\right)
 \end{aligned}$$

$$= \frac{a^3 b^2 c^2}{2520}$$

**Example:**

Find the value of  $\iiint xyz dx dy dz$  through the positive spherical octant for which  $x^2 + y^2 + z^2 \leq a^2$

**Solution:**

In the positive octant, the limits are

$z$  varies from 0 to  $\sqrt{a^2 - x^2 - y^2}$

$y$  varies from 0 to  $\sqrt{a^2 - x^2}$

$x$  varies from 0 to  $a$

$$\begin{aligned} I &= \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz dz dy dx \\ &= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[ \frac{xyz^2}{2} \right]_0^{\sqrt{a^2-x^2-y^2}} dy dx \\ &= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy(a^2 - x^2 - y^2) dy dx \\ &= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} (a^2 xy - x^3 y - xy^3) dy dx \\ &= \frac{1}{2} \int_0^a \left[ \frac{a^2 xy^2}{2} - \frac{x^3 y^2}{2} - \frac{xy^4}{4} \right]_0^{\sqrt{a^2-x^2}} dx \\ &= \frac{1}{2} \int_0^a \left( \frac{a^2 x(a^2-x^2)}{2} - \frac{x^3(a^2-x^2)}{2} - \frac{x(a^2-x^2)^2}{4} \right) dx \\ &= \frac{1}{2} \int_0^a \frac{x}{2} (a^2 - x^2) \left[ a^2 - x^2 - \frac{(a^2-x^2)}{2} \right] dx \\ &= \frac{1}{2} \int_0^a \frac{x(a^2-x^2)(a^2-x^2)}{4} dx \\ &= \frac{1}{8} \int_0^a x(a^2 - x^2)^2 dx \end{aligned}$$

$$\text{Put } a^2 - x^2 = t$$

$$x = 0 \rightarrow t = a^2$$

$$-2x dx = dt$$

$$x = a \rightarrow t = 0$$

$$\Rightarrow I = \frac{1}{8} \int_{a^2}^0 t^2 \left( -\frac{dt}{2} \right)$$

$$\begin{aligned}
&= -\frac{1}{16} \int_{a^2}^0 t^2 dt \\
&= \frac{1}{16} \int_0^{a^2} t^2 dt \\
&= \frac{1}{16} \left[ \frac{t^3}{3} \right]_0^{a^2} \\
&= \frac{1}{16} \left( \frac{a^6}{3} \right) \\
&= \frac{a^6}{48}
\end{aligned}$$

**Example:**

Evaluate  $\iiint_D (x + y + z) dx dy dz$  where  $D: 1 \leq x \leq 2, 2 \leq y \leq 3, 1 \leq z \leq 3$

**Solution:**

$$\begin{aligned}
\iiint_D (x + y + z) dx dy dz &= \int_1^2 \int_2^3 \int_1^3 (x + y + z) dz dy dx \\
&= \int_1^2 \int_2^3 \left[ xz + yz + \frac{z^2}{2} \right]_1^3 dy dx \\
&= \int_1^2 \int_2^3 \left( 3x + 3y + \frac{9}{2} - x - y - \frac{1}{2} \right) dy dx \\
&= \int_1^2 \int_2^3 (2x + 2y + 4) dy dx \\
&= \int_1^2 \left[ 2xy + \frac{2y^2}{2} + 4y \right]_2^3 dx \\
&= \int_1^2 (6x + 9 + 12 - 4x - 4 - 8) dx \\
&= \int_1^2 (2x + 9) dx \\
&= \left[ \frac{2x^2}{2} + 9x \right]_1^2 \\
&= 4 + 18 - 1 - 9 \\
&= 12
\end{aligned}$$

**Example:**

Evaluate  $\iiint \frac{dx dy dz}{\sqrt{a^2 - x^2 - y^2 - z^2}}$  over the first octant of the sphere  $x^2 + y^2 + z^2 = a^2$

**Solution:**

$$\begin{aligned}
 \iiint \frac{dx dy dz}{\sqrt{a^2 - x^2 - y^2 - z^2}} &= \int_0^a \int_0^{\sqrt{a^2 - x^2}} \int_0^{\sqrt{a^2 - x^2 - y^2}} \frac{dz dy dx}{\sqrt{a^2 - x^2 - y^2 - z^2}} \\
 &= \int_0^a \int_0^{\sqrt{a^2 - x^2}} \int_0^{\sqrt{a^2 - x^2 - y^2}} \frac{dz dy dx}{\sqrt{\sqrt{(a^2 - x^2 - y^2)^2 - z^2}}} \\
 &= \int_0^a \int_0^{\sqrt{a^2 - x^2}} \left[ \sin^{-1} \frac{z}{\sqrt{a^2 - x^2 - y^2}} \right]_0^{\sqrt{a^2 - x^2 - y^2}} dy dx \\
 &= \int_0^a \int_0^{\sqrt{a^2 - x^2}} (\sin^{-1} 1 - 0) dy dx \\
 &= \int_0^a \int_0^{\sqrt{a^2 - x^2}} \frac{\pi}{2} dy dx \\
 &= \frac{\pi}{2} \int_0^a [y]_0^{\sqrt{a^2 - x^2}} dx \\
 &= \frac{\pi}{2} \int_0^a \sqrt{a^2 - x^2} dx \\
 &= \frac{\pi}{2} \left[ \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a \\
 &= \frac{\pi}{2} \left[ 0 + \frac{a^2}{2} \sin^{-1} 1 - 0 - 0 \right] \\
 &= \frac{\pi a^2}{2} \frac{\pi}{2} \\
 &= \frac{\pi^2 a^2}{8}
 \end{aligned}$$

### Exercise:

1. Evaluate  $\int_0^4 \int_0^1 \int_0^1 (x + y + z) dz dy dx$  **Ans: 12**
2. Evaluate  $\int_0^1 \int_0^{1-x} \int_0^{x+y} x dz dy dx$  **Ans:  $\frac{1}{8}$**
3. Evaluate  $\int_1^3 \int_{\frac{1}{x}}^1 \int_0^{\sqrt{xy}} x y z dz dy dx$  **Ans:  $\frac{2}{5} \left[ \frac{2}{5} (9\sqrt{3} - 1) - \log 3 \right]$**
4. Evaluate  $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z} dz dy dx$  **Ans:  $\frac{5}{8}$**
5. Evaluate  $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} dz dy dx$  **Ans:  $\frac{1}{8} [e^{4a} - 6e^{2a} + 8e^a - 3]$**
6. Evaluate  $\iiint_V (x + y + z) dx dy dz$  where the region V is bounded by



$$x + y + z = a \ (a > 0), x = 0, y = 0, z = 0 \quad \text{Ans: } \frac{a^4}{8}$$

$$7. \text{ Evaluate } \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{\sqrt{1-x^2-y^2-z^2}} \quad \text{Ans: } \frac{\pi^2}{8}$$

$$8. \text{ Evaluate } \int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x + y + z) dx dy dz \quad \text{Ans: } 0$$

### Triple Integrals – Volume of Solids

Volume =  $\iiint_V dzdydx$  where V is the volume of the given surface.

**Example:**

**Find the volume of the sphere  $x^2 + y^2 + z^2 = a^2$**

**Solution:**

Volume = 8 X volume of the first octant

z varies from 0 to  $\sqrt{a^2 - x^2 - y^2}$

y varies from 0 to  $\sqrt{a^2 - x^2}$

x varies from 0 to a

$$\begin{aligned} &= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dzdydx \\ &= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} [Z]_0^{\sqrt{a^2-x^2-y^2}} dydx \\ &= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2} dydx \\ &= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{(\sqrt{a^2 - x^2})^2 - y^2} dydx \\ &= 8 \int_0^a \left[ \frac{y}{2} \sqrt{a^2 - x^2 - y^2} + \frac{a^2 - x^2}{2} \sin^{-1} \frac{y}{\sqrt{a^2 - x^2}} \right]_0^{\sqrt{a^2 - x^2}} dx \\ &= 8 \int_0^a \left( 0 + \frac{(a^2 - x^2)}{2} \sin^{-1} 1 - 0 \right) dx \\ &= 4 \int_0^a (a^2 - x^2) \frac{\pi}{2} dx \\ &= 2\pi \int_0^a (a^2 - x^2) dx \\ &= 2\pi \left[ a^2x - \frac{x^3}{3} \right]_0^a \end{aligned}$$

$$= 2\pi \times \frac{2a^3}{3}$$

$$= \frac{4\pi a^3}{3} \text{ cu. units.}$$

**Example:**

**Find the volume of the ellipsoid  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$**

**Solution:**

Volume = 8 X volume of the first octant

z varies from 0 to  $c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$

y varies from 0 to  $\sqrt{1 - \frac{x^2}{a^2}}$

x varies from 0 to a

$$V = 8 \int_0^a \int_0^{\sqrt{1 - \frac{x^2}{a^2}}} \int_0^{c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} dz dy dx$$

$$= 8 \int_0^a \int_0^{\sqrt{1 - \frac{x^2}{a^2}}} [Z]_0^{c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} dy dx$$

$$= 8 \int_0^a \int_0^{\sqrt{1 - \frac{x^2}{a^2}}} c\sqrt{\left(1 - \frac{x^2}{a^2}\right) - \frac{y^2}{b^2}} dy dx$$

$$= 8c \int_0^a \int_0^{\sqrt{1 - \frac{x^2}{a^2}}} \sqrt{\frac{b^2\left(1 - \frac{x^2}{a^2}\right) - y^2}{b^2}} dy dx$$

$$= \frac{8c}{b} \int_0^a \int_0^{\sqrt{1 - \frac{x^2}{a^2}}} \sqrt{b^2\left(1 - \frac{x^2}{a^2}\right) - y^2} dy dx$$

$$= \frac{8c}{b} \int_0^a \int_0^k \sqrt{k^2 - y^2} dy dx \quad \text{where } k^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$= \frac{8c}{b} \int_0^a \left[ \frac{y}{2} \sqrt{k^2 - y^2} + \frac{k^2}{2} \sin^{-1} \frac{y}{k} \right]_0^k dx$$

$$= \frac{8c}{b} \int_0^a \left( 0 + \frac{k^2}{2} \sin^{-1} 1 - 0 \right) dx$$

$$= \frac{8c}{b} \int_0^a \left( \frac{k^2}{2} \right) \frac{\pi}{2} dx$$

$$\begin{aligned}
&= \frac{2c\pi}{b} \int_0^a k^2 dx \\
&= \frac{2c\pi}{b} \int_0^a b^2 \left(1 - \frac{x^2}{a^2}\right) dx \\
&= 2bc\pi \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx \\
&= 2bc\pi \left[ x - \frac{x^3}{3a^2} \right]_0^a \\
&= 2bc\pi \left[ a - \frac{a^3}{3a^2} \right] \\
&= 2bc\pi \left( a - \frac{a}{3} \right) \\
&= 2bc\pi \times \frac{2a}{3} \\
&= \frac{4\pi abc}{3} \text{ cu. units.}
\end{aligned}$$

**Example:**

**Find the volume of the tetrahedron bounded by the coordinate planes and**

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

**Solution:**

$$\text{Volume} = \iiint_V dzdydx$$

$$z \text{ varies from } 0 \text{ to } c \left(1 - \frac{x}{a} - \frac{y}{b}\right)$$

$$y \text{ varies from } 0 \text{ to } b \left(1 - \frac{x}{a}\right)$$

$$x \text{ varies from } 0 \text{ to } a$$

$$\begin{aligned}
V &= \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dzdydx \\
&= \int_0^a \int_0^{b(1-\frac{x}{a})} [Z]_0^{c(1-\frac{x}{a}-\frac{y}{b})} dydx \\
&= \int_0^a \int_0^{b(1-\frac{x}{a})} c \left(1 - \frac{x}{a} - \frac{y}{b}\right) dydx \\
&= c \int_0^a \left[ y - \frac{xy}{a} - \frac{y^2}{2b} \right]_0^{b(1-\frac{x}{a})} dx
\end{aligned}$$

$$\begin{aligned}
 &= c \int_0^a b \left[ \left(1 - \frac{x}{a}\right) - \frac{x}{a} \left(1 - \frac{x}{a}\right) - \frac{b^2 \left(1 - \frac{x}{a}\right)^2}{2b} \right] dx \\
 &= bc \int_0^a \left[ \left(1 - \frac{x}{a}\right)^2 - \frac{1}{2} \left(1 - \frac{x}{a}\right)^2 \right] dx \\
 &= \frac{bc}{2} \int_0^a \left(1 - \frac{x}{a}\right)^2 dx \\
 &= \frac{bc}{2} \left[ \frac{\left(1 - \frac{x}{a}\right)^3}{3 \left(-\frac{1}{a}\right)} \right]_0^a \\
 &= -\frac{abc}{6} \left[ \left(1 - \frac{x}{a}\right)^3 \right]_0^a \\
 &= -\frac{abc}{6} (0 - 1) \\
 &= \frac{abc}{6} \text{ cu. units.}
 \end{aligned}$$

**Example:**

Evaluate  $\iiint_V dx dy dz$  where  $V$  is the volume enclosed by the cylinder  $x^2 + y^2 = 1$  and the planes  $z = 0$  and  $z = 2 - x$ .

**Solution:**

In the positive octant, the limits are

$z$  varies from 0 to  $2 - x$

$x$  varies from 0 to  $\sqrt{1 - y^2}$

$y$  varies from  $-1$  to  $1$

$$\begin{aligned}
 \iiint dx dy dz &= 2 \int_{-1}^1 \int_0^{\sqrt{1-y^2}} \int_0^{2-x} dz dx dy \\
 &= 2 \int_{-1}^1 \int_0^{\sqrt{1-y^2}} [z]_0^{2-x} dx dy \\
 &= 2 \int_{-1}^1 \int_0^{\sqrt{1-y^2}} (2 - x) dx dy \\
 &= 2 \int_{-1}^1 \left[ 2x - \frac{x^2}{2} \right]_0^{\sqrt{1-y^2}} dy \\
 &= 2 \int_{-1}^1 \left[ 2\sqrt{1-y^2} - \left(\frac{1-y^2}{2}\right) \right] dy
 \end{aligned}$$

$$\begin{aligned}
 &= 4 \int_{-1}^1 [\sqrt{1-y^2}] dy - \int_{-1}^1 [1-y^2] dy \\
 &= 4 \left[ \frac{y}{2} \sqrt{1-y^2} + \frac{1}{2} \sin^{-1} y \right]_{-1}^1 - \left[ y - \frac{y^3}{3} \right]_{-1}^1 \\
 &= 4 \left[ 0 + \frac{1}{2} \sin^{-1} 1 - 0 - \frac{1}{2} \sin^{-1}(-1) \right] - \left[ 1 - \frac{1}{3} + 1 - \frac{1}{3} \right] \\
 &= 4 \left[ \frac{1}{2} \frac{\pi}{2} + \frac{1}{2} \frac{\pi}{2} \right] - \left[ 2 - \frac{2}{3} \right] \\
 &= 4 \left( \frac{2\pi}{4} \right) - \frac{4}{3} \\
 &= 2\pi - \frac{4}{3}
 \end{aligned}$$

### Exercise:

1. Find the volume of the tetrahedron whose vertices are (0,0,0), (0,1,0), (1,0,0) and (0,0,1)

**Ans:**  $\frac{1}{6}$  cu. units.

2. Evaluate  $\iiint_V dz dx dy$ , where V is the volume enclosed by the cylinder  $x^2 + y^2 = 4$  and the planes  $y + z = 4$  and  $z = 0$ .

**Ans:**  $16\pi$  cu. units

3. Find the volume of the region bounded by the paraboloid  $z = x^2 + y^2$  and the plane  $z = 4$

**Ans:**  $8\pi$  cu. units

4. Find the volume of the ellipsoid  $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$  by using triple integration.

**Ans:**  $32\pi$  cu. units

5. Find the volume of the tetrahedron bounded by coordinate planes and the plane

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$$

**Ans:** 4 cu. units