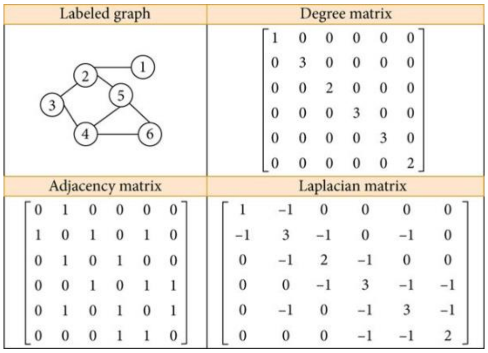
# 2.6 Graph Laplacians




Graph Laplacians are core mathematical tools used in graph theory, data science, machine learning, image processing, and network analysis. They help us understand connectivity, flow, clustering, and smoothness on graphs.

**1. What is a Graph Laplacian?**

A Graph Laplacian is a matrix representation of a graph that captures:

- How nodes are connected
- How information diffuses across the graph
- How clusters (communities) are formed

📌 It is derived from:

- **Adjacency Matrix (A)**
- **Degree Matrix (D)**

**2. Basic Components**

**(a) Adjacency Matrix (A)**

- $A_{ij} = 1$if node $i$ is connected to node $j$
- $A_{ij} = 0$otherwise

Example:

A = [0 1 1

1 0 0

1 0 0]

**(b) Degree Matrix (D)**

- Diagonal matrix
- $D_{ii}$= number of edges connected to node $i$

Example:

D = [2 0 0

0 1 0

0 0 1]

**Types**

1. **Unnormalized Laplacian**:

$$L = D - A$$

2. **Normalized Laplacian**:
   - $L_{sym} = I - D^{-1/2}AD^{-1/2}$
   - $L_{rw} = I - D^{-1}A$

**Unnormalized Graph Laplacian**

**Definition:**

$$L = D - A$$

**Example:**

L = [ 2 -1 -1

-1  1  0

-1  0  1]

**Key Properties:**

✓ Symmetric
✓ Positive semi-definite
✓ Small eigenvalues → strong connectivity
✓ Zero eigenvalue → number of connected components

**Interpretation:**
Measures how much a node differs from its neighbors.

## Normalized Graph Laplacians

Normalization helps when node degrees vary widely.

### (a) Symmetric Normalized Laplacian

$$L_{sym} = I - D^{-1/2} A D^{-1/2}$$

✓ Used in spectral clustering
✓ Eigenvalues lie between 0 and 2

### (b) Random Walk Laplacian

$$L_{rw} = I - D^{-1} A$$

✓ Related to Markov chains & random walks
✓ Used in PageRank and diffusion models

### Intuition Behind Laplacian

Think of the graph as a **rubber sheet**:

- Connected nodes try to stay close
- Laplacian measures tension
- Minimizing Laplacian energy → smooth labels or clusters

Mathematically:

$$x^T L x = \sum_{(i,j) \in E} (x_i - x_j)^2$$

📌 If neighbors have similar values → small energy

### 6. Eigenvalues & Eigenvectors (Very Important)

| **Eigenvalue** | **Meaning** |
|---|---|
| 0 | Graph is connected |
| Multiple 0s | Multiple components |

| | |
|---|---|
| Small values | Strong clusters |
| Eigenvectors | Reveal community structure |

➞ Used in **Spectral Clustering**

**7. Applications of Graph Laplacians**

◇ **Data Science & ML**

- Spectral clustering
- Semi-supervised learning
- Dimensionality reduction (Laplacian Eigenmaps)

◇ **Network Analysis**

- Community detection
- Centrality measures
- Graph partitioning

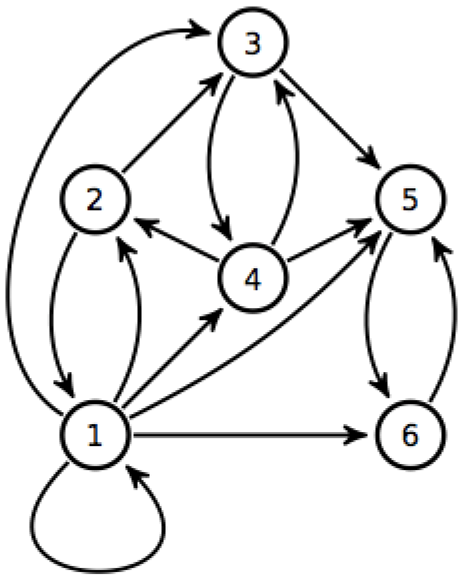
◇ **Image Processing**

- Image segmentation
- Edge detection

◇ **Physics & Engineering**

- Heat diffusion
- Electrical networks

## PageRank Algorithm




PageRank is an algorithm used to rank web pages (nodes) based on their importance in a directed graph.

**Core Idea**

A page is important if many important pages link to it.

**Formula**

$$PR(i) = \frac{1-d}{N} + d \sum_{j \in In(i)} \frac{PR(j)}{Out(j)}$$

Where:

- $PR(i)$→ PageRank of page $i$
- $d$→ damping factor (usually **0.85**)
- $N$→ total number of pages

- $In(i)$→ pages linking to $i$
- $Out(j)$→ outgoing links from page $j$

**Random Surfer Model**

- User follows links with probability **d**
- Jumps to a random page with probability $\mathbf{1 - d}$

**Steps**

1. Initialize all pages with equal rank
2. Update ranks using the formula
3. Repeat until values converge

**Properties**

- Works on **directed graphs**
- Handles dead ends using damping factor
- Converges to a stable ranking

**Applications**

- Web search engines
- Social network influence
- Citation networks