DISCRETE TIME FOURIER TRANSFORM AND ITS PROPERTIES

The DTFT is a transformation that maps Discrete-time (DT) signal x[n] into a complex valued function of the real variable namely:

$$F[x(n)] = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

INVERSE DISCRETE FOURIER TRANSFORM

IDTFT is given by

$$x(n) = F^{-1}[X(e^{j\omega})] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\, d\omega \text{ , for } n=-\infty \text{ to } \infty$$

Example 1: Find the DTFT of $\boldsymbol{x(n)=a^n u(n)}$.

Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n u[n]e^{-j\omega n} = \sum_{n=0}^{\infty}\left(ae^{-j\omega}\right)^{-n} = \frac{1}{1-ae^{-j\omega}}.$$

Example 2: Find the DTFT of $\boldsymbol{x(n) = a^{|n|}}$, $\boldsymbol{|a|<1}$.

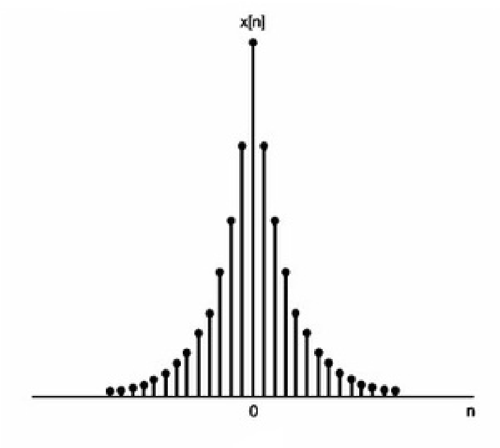
Solution:

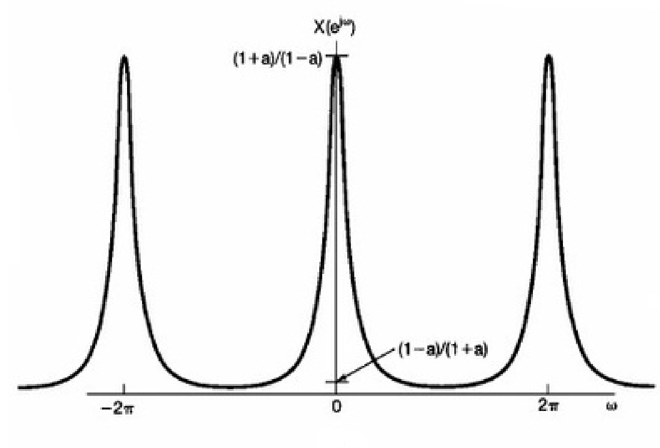
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|}u[n]e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n}e^{-j\omega n} + \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

Let $m=-n$ in the first summation, we obtain

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|}u[n]e^{-j\omega n} = \sum_{m=1}^{\infty} a^m e^{j\omega m} + \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \frac{ae^{j\omega}}{1-ae^{j\omega}} + \frac{1}{1-ae^{-j\omega}} = \frac{1-a^2}{1-2a\cos\omega + a^2}$$





PROPERTIES OF DTFT:

Linearity:

$$\mathcal{F}_{DT}\{x_1(n) + x_2(n)\} = \mathcal{F}_{DT}\{x_1(n)\} + \mathcal{F}_{DT}\{x_2(n) = X_1(e^{j\omega}) + X_2(e^{j\omega})$$

Time Shifting:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\{x(n - n_0)\} = e^{-j\omega n_0}\, X(e^{j\omega})$$

Proof:

$$\sum_{n=-\infty}^{\infty} x(n-n_0)e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x(m)e^{-j\omega(m+n_0)} = e^{-j\omega n_0}\sum_{m=-\infty}^{\infty} x(m)e^{-j\omega m}$$

Time Reversal:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\{x(-n)\} = X(e^{-j\omega})$$

Convolution:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$ and $\mathcal{F}_{DT}\{y(n)\} = Y(e^{j\omega})$, then:

$$G(e^{j\omega}) = \mathcal{F}_{DT}\{x(n) * y(n)\} = X(e^{j\omega})\, Y(e^{j\omega})$$

Proof:

$$G\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}x\left(m\right)y\left(n-m\right)e^{-jn\omega}=\sum_{n=-\infty}^{\infty}x\left(m\right)\sum_{m=-\infty}^{\infty}y\left(n-m\right)e^{-jn\omega}$$

$$=\sum_{m=-\infty}^{\infty}x\left(m\right)\sum_{r=-\infty}^{\infty}y\left(r\right)e^{-j\left(m+r\right)\omega}=\sum_{m=-\infty}^{\infty}x\left(m\right)e^{-jm\omega}\sum_{r=-\infty}^{\infty}y\left(r\right)e^{-jr\omega}$$

Frequency Shifting:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\left\{e^{j\omega_0 n}x\left(n\right)\right\}=X\left(e^{j\left(\omega-\omega_0\right)}\right),$$

Time Multiplication:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\left\{nx\left(n\right)\right\}=-z\frac{dX\left(z\right)}{dz}\bigg|_{z=e^{j\omega}}$$

Parseval's Theorem:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$ and $\mathcal{F}_{DT}\{y(n)\} = Y(e^{j\omega})$, then:

$$\sum_{n=-\infty}^{\infty}x\left(n\right)y^*\left(n\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X\left(e^{j\omega}\right)Y^*\left(e^{j\omega}\right)d\omega$$

Proof:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty}x(n)e^{-jn\omega}\right]Y*\left(e^{j\omega}\right)d\omega=\sum_{n=-\infty}^{\infty}x(n)\frac{1}{2\pi}\int_{-\pi}^{\pi}Y*\left(e^{j\omega}\right)e^{-j\omega}d\omega$$

For the case $x(n) = y(n)$, then:

$$\sum_{n=-\infty}^{\infty}\left|x(n)\right|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

Multiplication of Sequences:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$ and $\mathcal{F}_{DT}\{y(n)\} = Y(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\left\{x(n)y(n)\right\}=\sum_{n=-\infty}^{\infty}x(n)y(n)e^{-j\omega n}=\sum_{n=-\infty}^{\infty}x(n)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}Y\left(e^{j\lambda}\right)e^{j\lambda n}d\lambda\right]e^{-j\omega n}$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}Y\left(e^{j\lambda}\right)d\lambda\sum_{n=-\infty}^{\infty}x(n)e^{-j(\omega-\lambda)n}=\frac{1}{2\pi}\int_{-\pi}^{\pi}Y\left(e^{j\lambda}\right)X\left(e^{j(\omega-\lambda)}\right)d\lambda$$

$$=\frac{1}{2\pi}Y\left(e^{j\omega}\right)*X\left(e^{j\omega}\right)$$

Differentiationinthe Frequency Domain:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\frac{dX\left(e^{j\omega}\right)}{d\omega}=\frac{d}{d\omega}\left[\sum_{n=-\infty}^{\infty}x(n)e^{-j\omega n}\right]=-j\sum_{n=-\infty}^{\infty}nx(n)e^{-j\omega n}=-j\mathcal{F}_{DT}\left\{nx(n)\right\}$$