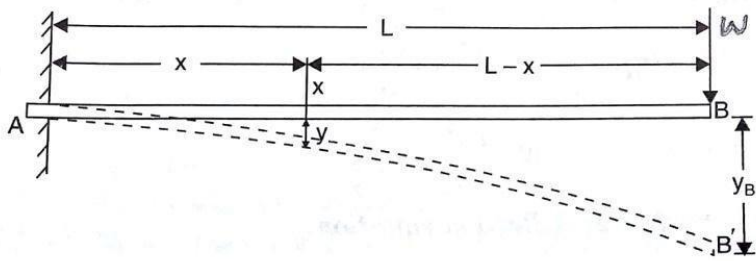


### 3.2 DOUBLE INTEGRATION METHOD

#### 3.2.1. DEFLECTION OF CANTILEVER WITH A POINT LOAD AT THE FREE END BY DOUBLE INTEGRATION METHOD

A cantilever AB of length L fixed at the point A and free end at the point B and carrying a point load at the free end B as shown in fig. AB shows the position of cantilever before any load is applied whereas AB' shows the position of cantilever after loading.



Consider a section X, at a distance x from the fixed end A. The B.M. at this section is given by,

$$M_x = -W(L-x) \quad (\text{minus sign due to hogging})$$

But B.M at any section is also given by

$$M = EI \frac{d^2 y}{dx^2}$$

Equating the two values of B.M., we get

$$EI \frac{d^2 y}{dx^2} = -W(L-x) = -WL + Wx$$

Integrating the above equation, we get

$$EI \frac{dy}{dx} = -WLx + \frac{Wx^2}{2} + C_1 \quad \dots(i)$$

Integrating again, we get

$$EI y = -\frac{WLx^2}{2} + \frac{W}{2} X \frac{x^3}{3} + C_1 x + C_2 \dots(ii)$$

Where C1 and C2 are constant of integration. Their values are obtained from boundary

conditions, which are: (i) at x=0, y=0 (ii) x=0,  $\frac{dy}{dx} = 0$

(At the fixed end, deflection and slopes are zero)

(i) By substituting  $x=0, y=0$  in equation (ii), we get  $0=0+0+0+C_2 \therefore C_2 = 0$

By substituting  $x=0, \frac{dy}{dx} = 0$  in equation (i), we get  
 $0=0+0+C_1 \therefore C_1 = 0$

Substituting the value of  $C_1$  in equation (i), we get

$$\begin{aligned} EI \frac{dy}{dx} &= -WLx + \frac{Wx^2}{2} \\ &= -W \left[ Lx - \frac{x^2}{2} \right] \dots (iii) \end{aligned}$$

Equation (iii) is known as slope equation. We can find the slope at any point on the cantilever by substituting the value of  $x$ . The slope and deflection are maximum at the free end. These can be determined by substituting  $x=L$  in these equations. Substituting the values of  $C_1$  and  $C_2$  in equation (ii), we get

$$\begin{aligned} EIy &= -WL \frac{x^2}{2} + \frac{Wx^3}{6} \\ &= -W \left[ \frac{Lx^2}{2} - \frac{x^3}{6} \right] \dots (iv) \end{aligned}$$

Equation (iv) is known as deflection equation.

Let  $\theta_B$  = slope at free end B  $y_B$  = Deflection at the free end B

Substituting  $\theta_B$  for  $\frac{dy}{dx}$  and  $x=L$  in equation (iii), we get

$$\begin{aligned} EI\theta_B &= -W \left[ L \cdot L - \frac{L^2}{2} \right] = -W \frac{L^2}{2} \\ \therefore \theta_B &= -\frac{WL^2}{2EI} \end{aligned}$$

Negative sign shows the tangent at B makes an angle in the anti-clockwise direction with AB.

$$\therefore \theta_B = \frac{WL^2}{2EI}$$

Substituting  $y_B$  for  $y$  and  $x=L$  in equation (iv), we get

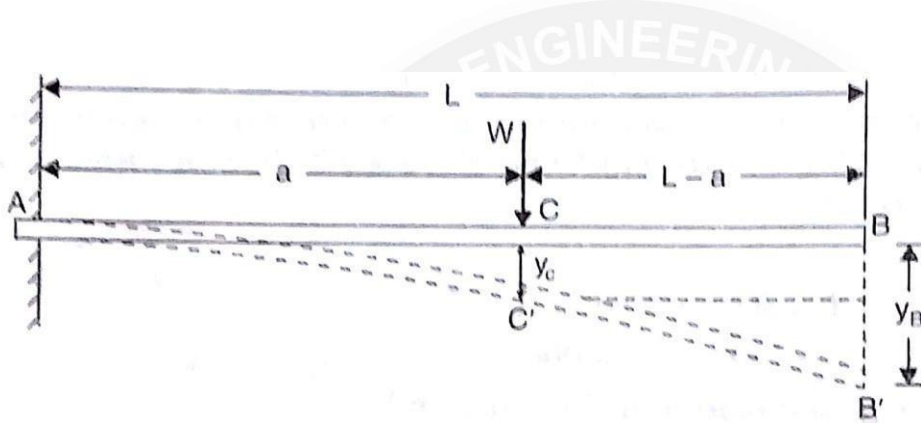
$$\begin{aligned} EIy_B &= -W \left[ L \cdot \frac{L^2}{2} - \frac{L^3}{6} \right] \\ &= -W \left[ \frac{L^3}{2} - \frac{L^3}{6} \right] \\ &= -W \cdot \frac{L^3}{3} \end{aligned}$$

$$\therefore y_B = -\frac{WL^3}{3EI} \text{ (Negative sign shows that deflection is downwards)}$$

$$\therefore y_B = \frac{WL^3}{3EI}$$

### 3.2.2 DEFLECTION OF A CANTILEVER WITH A POINT LOAD AT A DISTANCE 'a' FROM THE FIXED END

A cantilever AB of length L fixed at point B and carrying a point load W at a distance 'a' from the fixed end A, is shown in Fig.



Let  $\theta_C = \text{Slope at point C i.e., } \frac{dy}{dx} \text{ at C}$   
 $y_C = \text{Deflection at point C}$   
 $y_B = \text{Deflection at point B}$

The portion AC of the cantilever may be taken as similar to a cantilever in Art. (i.e., load at the free end).

$$\therefore \theta_C = +\frac{Wa^2}{2EI}$$

$$\text{and } y_C = \frac{Wa^3}{3EI}$$

The beam will bend only between A and C, but from C to B, it will remain straight since B.M. between C and B is zero.

Since the portion CB of the cantilever is straight, therefore Slope at C = Slope at B

$$\theta_C = \theta_B = \frac{Wa^2}{2EI}$$

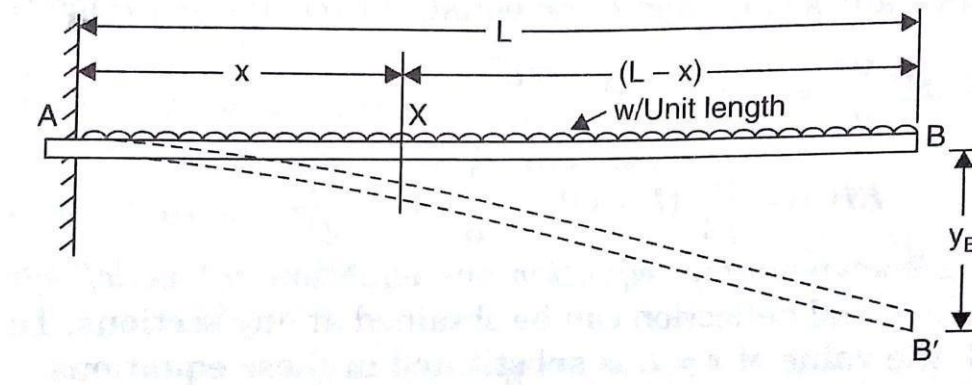
Now from Fig. we have

$$y_B = y_C + \theta_C(L-a)$$

$$= \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(L-a) \left[ \text{since, } \theta_C = \frac{Wa^2}{2EI} \right]$$

### 3.2.3.DEFLECTION OF A CANTILEVER WITH A UNIFORMLY DISTRIBUTED LOAD

A cantilever AB of length L fixed at the point A and free at the point B and carrying a uniformly distributed load of w per unit length, is shown in Fig.



Consider a section X, at a distance x from the fixed end A. The B.M. at this section is given by,

$$M_x = -w(L-x) \cdot \frac{(L-x)}{2} \text{ (Minus sign due to hogging)}$$

But B.M. at any section is also given by equation as

$$M = EI \frac{d^2y}{dx^2}$$

Equating the two values of B.M., we get

$$EI \frac{d^2y}{dx^2} = -\frac{w}{2}(L-x)^2$$

Integrating the above equation, we get

$$\begin{aligned} EI \frac{dy}{dx} &= -\frac{w}{2} \frac{(L-x)^3}{3} (-1) + C_1 \\ &= \frac{w}{6}(L-x)^3 + C_1 \dots\dots\dots (i) \end{aligned}$$

Integrating again, we get

$$\begin{aligned} EIy &= \frac{w}{6} \frac{(L-x)^4}{4} (-1) + C_1x + C_2 \\ &= -\frac{w}{24}(L-x)^4 + C_1x + C_2 \dots\dots\dots (ii) \end{aligned}$$

where  $C_1$  and  $C_2$  are constant of integrations. Their values are obtained from boundary conditions, which are : (i) at  $x = 0$ ,  $y = 0$  and (ii) at  $x = 0$ ,  $\frac{dy}{dx} = 0$  (as the deflection and slope at fixed end A are zero).

(i) By substituting  $x = 0$ ,  $y = 0$  in equation (ii), we get

$$0 = -\frac{w}{24}(L - 0)^4 + C_1 \times 0 + C_2 = -\frac{wL^4}{24} + C_2$$

$$\therefore C_2 = \frac{wL^4}{24}$$

(ii) By substituting  $x = 0$  and  $\frac{dy}{dx} = 0$  in equation (i), we get

$$= \frac{w}{64}(L - 0)^3 + C_1 = -\frac{wL^3}{6} + C_1$$

$$\therefore C_1 = \frac{wL^3}{6}$$

Substituting the values of  $C_1$  and  $C_2$  in equations (i) and (ii), we get

$$EI \frac{dy}{dx} = \frac{w}{6}(L - x)^3 - \frac{wL^3}{6} \dots (iii) \text{ and } EIy = -\frac{w}{24}(L - x)^4 - \frac{wL^3}{6}x + \frac{wL^4}{24} \dots (iv)$$

Equation (iii) is known as slope equation and equation (iv) as deflection equation. From these equations the slope and deflection can be obtained at any sections. To find the slope and deflection at point B, the value of  $x = L$  is substituted in these equations. Let

$\theta_B =$  Slope at the free end B i.e.,  $\frac{dy}{dx}$  at B  $y_B =$  Deflection at the free end B.

From equation (iii), we get slope at B as

$$EI \theta_B = \frac{w}{6}(L - L)^3 - \frac{wL^3}{6} = -\frac{wL^3}{6}$$

$$\theta_B = -\frac{wL^3}{6EI} = -\frac{WL^2}{6EI} \quad (\text{Since, } W = \text{Total load} = w.L)$$

From equation (iv), we get the deflection at B as

$$EI y_B = -\frac{w}{24}(L - L)^4 - \frac{wL^3}{6} \times L + \frac{wL^4}{24}$$

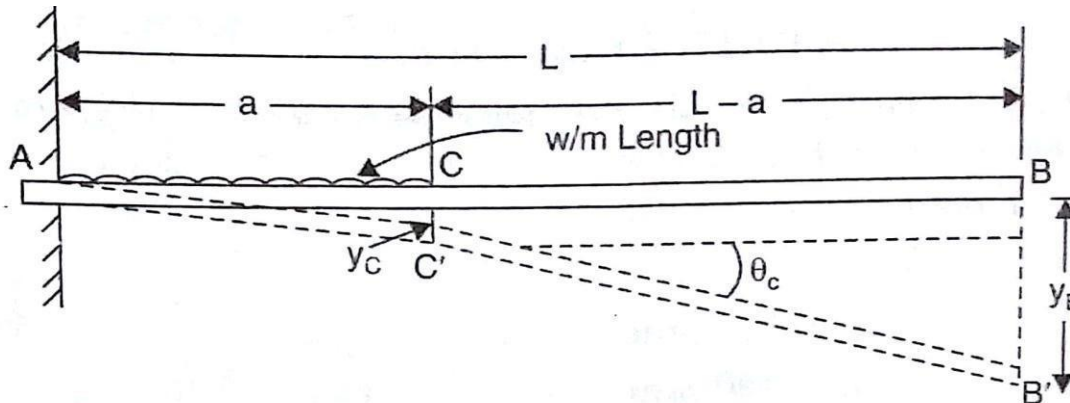
$$= -\frac{wL^4}{6} + \frac{wL^4}{24} = -\frac{3}{24}wL^4 = -\frac{wL^4}{8}$$

$$\therefore y_B = -\frac{wL^4}{8EI} = -\frac{WL^3}{8EI} \quad (\text{Since, } W = w.L)$$

$$\therefore \text{Downward deflection at B, } y_B = -\frac{wL^4}{8EI}$$

### 3.2.4.DEFLECTION OF A CANTILEVER WITH A UNIFORMLY DISTRIBUTED LOAD FOR A DISTANCE 'a' FROM THE FIXED END

A cantilever AB of length L fixed at the point A and free at the point B and carrying a uniformly distributed load of w/m length for a distance 'a' from the fixed end, is shown in Fig.



The beam will bend only between A and C, but from C to B it will remain straight since B.M. between C and B is zero. The deflected shape of the cantilever is shown by AC'B' in which portion C'B' is straight.

Let  $\theta_C$  = Slope at C, i.e.,  $\frac{dy}{dx}$  at C  $y_C$  = Deflection at point C, and  $y_B$  = Deflection at point B.

The portion AC of the cantilever may be taken as similar to a cantilever in Art.

$$\therefore \theta_C = \frac{w \cdot a^2}{6EI}$$

$$\text{4 and } y_C = \frac{w \cdot a^3}{8EI}$$

Since the portion C'B' of the cantilever is straight, therefore slope at C = Slope at B

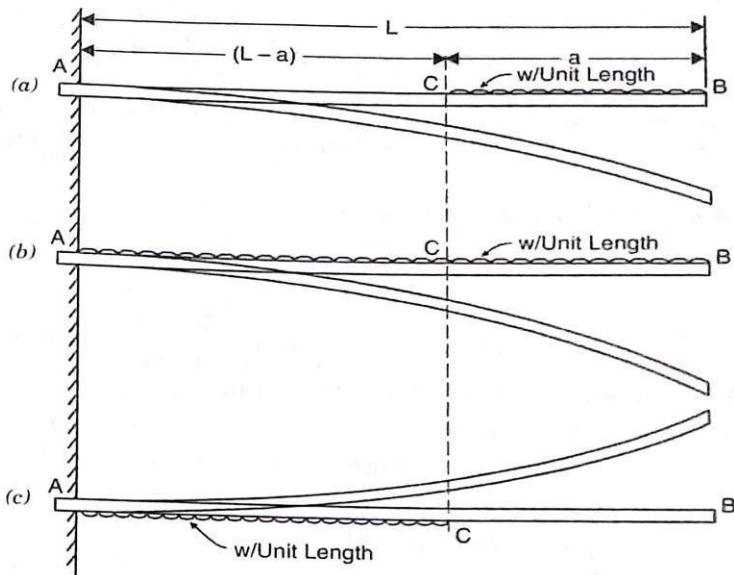
$$\text{or } \theta_C = \theta_B = \frac{w a^3}{6EI} \dots (1)$$

Now from Fig. we have

$$\begin{aligned} y_B &= y_C + \theta_C (L - a) \\ &= \frac{w a^4}{8EI} + \frac{w \cdot a^3}{6EI} (L - a) \end{aligned} \dots (2)$$

### 3.2.5.DEFLECTION OF A CANTILEVER WITH A UNIFORMLY DISTRIBUTED LOAD FOR A DISTANCE 'a' FROM THE FREE END

A cantilever AB of length L fixed at the point A and free at the point B and carrying a uniformly distributed load of w/m length for a distance 'a' from the free end, is shown in Fig.



The slope and deflection at the point B is determined by considering :

- (i) the whole cantilever AB loaded with a uniformly distributed load of w per unit length as shown in Fig.
- (ii) a part of cantilever from A to C of length (L - a) loaded with an upward uniformly distributed load of w per unit length as shown in Fig.

Then slope at B = Slope due to downward uniform load over the whole length

- Slope due to upward uniform load from A to C

and deflection at B = Deflection due to downward uniform load over the whole length

- deflection due to upward uniform load from A to C.

(a) Now slope at B due to downward uniformly distributed load over the whole length

$$= \frac{wL^3}{6EI}$$

(b) slope at B or at C due to upward uniformly distributed load over the length(L - a)

$$= \frac{w(L-a)^3}{6EI}$$

Hence net slope at B is given by,

$$\theta_B = \frac{wL^3}{6EI} - \frac{w(L-a)^3}{6EI} \dots (i)$$

The downward deflection of point B due to downward distributed load over the whole length AB

$$= \frac{wL^4}{8EI}$$

The upward deflection of point B due to upward uniformly distributed load acting on the portion AC

$$= \text{upward deflection of C} + \text{slope at C} \times CB$$

$$= \frac{w(L-a)^4}{8EI} + \frac{w(L-a)^3}{6EI} \times a \quad (\text{since } CB = a)$$

∴ Net downward deflection of the free end B is given by

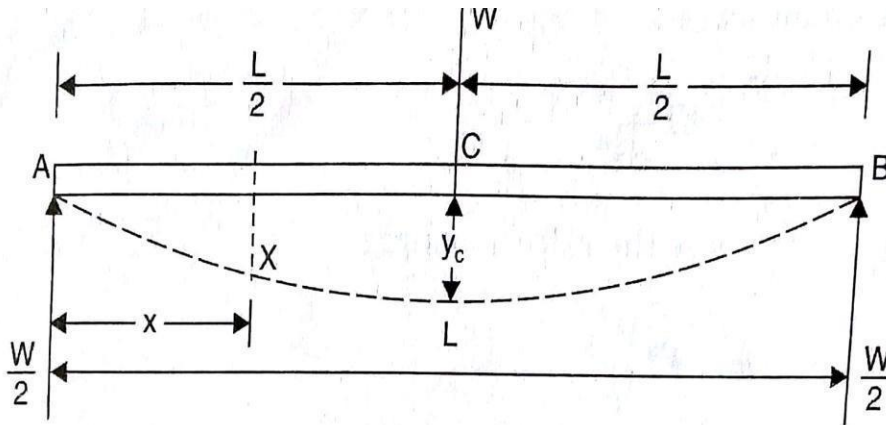
$$y_B = \frac{wL^4}{8EI} - \left[ \frac{w(L-a)^4}{8EI} + \frac{w(L-a)^3}{6EI} \times a \right] \dots (ii)$$

### 3.2.6.DEFLECTION OF A SIMPLY SUPPORTED BEAM CARRYING A POINT LOAD AT THE CENTRE

A simply supported beam AB of length L and carrying a point load W at the centre is shown in Fig.

As the load is symmetrically applied the reactions  $R_A$  and  $R_B$  will be equal. Also the maximum deflection will be at the centre.





Now  $R_A = R_B = \frac{W}{2}$

Consider a section X at a distance x from A. The bending moment at this section is given by,

$$M_x = R_A \times x$$

$$= \frac{W}{2} \times x$$

(Plus sign is as B.M. for left portion at X is clockwise)  
at any section is also given by equation as

But B.M.

$$M = EI \frac{d^2 y}{dx^2}$$

Equation the two values of B.M., we get

$$EI \frac{d^2 y}{dx^2} = \frac{W}{2} \times x \quad \dots (i)$$

On integration, we get

$$EI \frac{dy}{dx} = \frac{W}{2} \times \frac{x^2}{2} + C_1 \quad \dots (ii)$$

Where  $C_1$  is the constant of integration. And its value is obtained from boundary conditions. The boundary condition is that at  $x = \frac{L}{2}$ , slope  $\frac{dy}{dx} = 0$  (As the deflection is at the centre, hence slope at the centre will be zero). Substituting this boundary condition in equation (ii), we get

$$0 = \frac{W}{4} \times \left(\frac{L}{2}\right)^2 + C_1$$

$$\text{or } C_1 = -\frac{WL^2}{16}$$

Substituting the value of  $C_1$  in equation (ii), we get

$$2 \quad \frac{dy}{dx} = \frac{wx^2}{4} - \frac{WL}{16}$$

El..... (iii)

The above equation is known the slope equation. We can find the slope at any point on the beam by substituting the values of  $x$ . Slope is maximum at A. At A ,  $x = 0$  and hence slope at A will be obtained by substituting  $x = 0$  in equation (iii).

$$\therefore EI \frac{dy}{dx} = \frac{W}{4} \times 0 - \frac{WL^2}{16}$$

$$EI \times \theta_A = - \frac{WL^2}{16}$$

$$\therefore \theta_A = - \frac{WL^2}{16EI}$$

The slope at point B will be equal to  $\theta_A$ , since the load is symmetrically applied.

$$\therefore \theta_B = \theta_A = - \frac{WL^2}{16EI}$$

The above equation gives the slope in radians.

Deflection at any point

Deflection at any point is obtained by integrating the slope equation (iii). Hence integrating equation (iii), we get

$$2 \quad \frac{W}{4} \cdot \frac{x^3}{3} - \frac{WL}{16}x + C_2$$

El  $\times y = \dots$  (iv)

Where  $C_2$  is another constant of integration. At A,  $x = 0$  and the deflection ( $y$ ) is zero.

Hence substituting these values in equation (iv), we get

$$EI \times 0 = 0 - 0 + C_2$$

$$\text{Or } C_2 = 0$$

Substituting the values of  $C_2$  in equation (iv), we get ..... (v)

$$EI \times y = \frac{Wx^3}{12} - \frac{WL^2.x}{16}$$

The above equation is known as the deflection equation. We can find the deflection at any point on the beam by substituting the values of  $x$ . The deflection is maximum at centre point C, where  $x = \frac{L}{2}$ . Let  $y_c$  represents the deflection at C. Then substituting  $x = \frac{L}{2}$  and  $y = y_c$  in equation (iv), we get

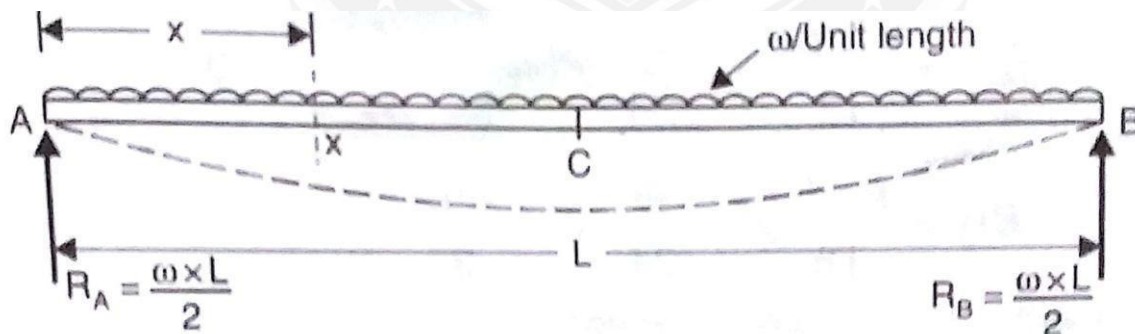
$$\begin{aligned}
 EI \times y_c &= \frac{w}{12} \left(\frac{L}{2}\right)^3 - \frac{wL^2}{16} \times \left(\frac{L}{2}\right) \\
 &= \frac{wL^3}{96} - \frac{wL^3}{32} = \frac{wL^3 - 3wL^3}{96} \\
 &= -\frac{2wL^3}{96} = -\frac{wL^3}{48} \\
 \therefore y_c &= -\frac{wL^3}{48EI}
 \end{aligned}$$

(Negative sign shows that deflection is downwards)

$$\therefore \text{Maximum deflection, } y_c = \frac{wL^3}{48EI}$$

### 3.2.7.DEFLECTION OF A SIMPLY SUPPORTED BEAM WITH A UNIFORMLY DISTRIBUTED LOAD

A simply supported beam AB of length L and carrying a uniformly distributed load of w per unit length is shown in Fig. The reactions at A and B will be equal. Also the maximum deflection will be at the centre. Each vertical reaction =  $\frac{w \times L}{2}$ .



$$\therefore R_A = R_B = \frac{w \times L}{2}$$

Consider a section X at a distance x from A. The bending moment at this section is given by,

$$M_x = R_A \times x - w \times \frac{x}{2} \times \frac{x}{2} = \frac{w \cdot L}{2} \times x - \frac{w \cdot x^2}{2}$$

But B.M. at any section is also given by equation ( ), as

$$M = EI \frac{d^2y}{dx^2}$$

Equation the two values of B.N., we get

$$EI \quad x \quad \frac{d^2y}{dx^2} = \frac{w \cdot L}{2} - w \cdot \frac{x^2}{2}$$

Integrating the above equation, we get

$$EI \frac{dy}{dx} = \frac{w \cdot L}{2} \cdot \frac{x^2}{2} - \frac{w}{2} \cdot \frac{x^3}{3} + C_1 \quad \dots(i)$$

where  $C_1$  is a constant of integration.

Integrating the above equation again, we get

$$EI \cdot y = \frac{w \cdot L}{4} \cdot \frac{x^3}{3} - \frac{w}{6} \cdot \frac{x^4}{4} + C_1 x + C_2 \quad \dots(ii)$$

where  $C_2$  is a another constant of integration. Thus two constant of integration (i.e.,  $C_1$  and  $C_2$ ) are obtained from boundary conditions. The boundary conditions are:

- (i) at  $x = 0, y = 0$  and (ii) at  $x = L, y = 0$

Substituting first boundary condition i.e.,  $x = 0, y = 0$  in equation (ii), we get

$$0 = 0 - 0 + 0 + C_2 \text{ or } C_2 = 0$$

Substituting the secondary boundary condition i.e.,  $x = L, y = 0$  in equation (ii), we get

$$\begin{aligned} 0 &= \frac{w \cdot L}{4} \cdot \frac{L^3}{3} - \frac{w}{6} \cdot \frac{L^4}{4} + C_1 \cdot L \quad (C_2 \text{ is already zero}) \\ &= \frac{w \cdot L^4}{12} - \frac{w \cdot L^4}{24} + C_1 \cdot L \\ \text{or } C_1 &= -\frac{wL^3}{12} + \frac{wL^3}{24} = -\frac{wL^3}{24} \end{aligned}$$

Substituting the value of  $C_1$  in equation (i) and (ii), we get

$$EI \frac{dy}{dx} = \frac{w \cdot L}{4} \cdot \frac{x^2}{2} - \frac{w}{6} \cdot \frac{x^3}{3} - \frac{wL^3}{24} \quad \dots(iii)$$

$$EI \cdot y = \frac{w \cdot L}{12} \cdot \frac{x^3}{3} - \frac{w}{24} \cdot \frac{x^4}{4} + \left[ -\frac{wL^3}{24} \right] x + 0 \quad (\text{since, } C_2 = 0)$$

$$EI y = \frac{w \cdot L}{12} \cdot \frac{x^3}{3} - \frac{w}{24} \cdot \frac{x^4}{4} - \frac{wL^3}{24} x \quad \dots(iv)$$

Equation (iii) is known as slope equation. We can find the slope

[i.e., the value of  $\frac{dy}{dx}$ ] at any point on the beam by substituting the different values of  $x$  in this equation.

Equation (iv) is known as deflection equation. We can find the deflection [*i. e. , the value of y*] at any point on the beam by substituting the different values of  $x$  in this equation.

Slope at the supports

Let  $\theta_A$  = Slope at support A.

and  $\theta_B$  = Slope at support B

At A,  $x = 0$  and  $\frac{dy}{dx} = \theta_A$ ,

Substituting these value in equation (iii), we get,

$$EI \theta_A = \frac{WL}{4} \times 0 - \frac{w}{6} \times 0 - \frac{WL^3}{24}$$

$$EI \times \theta_A = -\frac{wL^3}{24}$$

$$\therefore \theta_A = -\frac{WL^3}{24EI}$$

The slope at point B will be equal to  $\theta_A$ , since the load is symmetrically applied.

$$\therefore \theta_B = \theta_A = -\frac{WL^3}{24EI}$$

The above equation gives the slope in radians.

### Deflection at any point

The deflection is maximum at centre point C, where  $x = \frac{L}{2}$ . Let  $y_c$  represents the deflection at C. Then substituting  $x = \frac{L}{2}$  and  $y = y_c$  in equation (iv), we get

$$EI \times y_c = \frac{WL}{12} \left(\frac{L}{2}\right)^3 - \frac{w}{24} \left(\frac{L}{2}\right)^4 - \frac{WL^3}{24} \times \left(\frac{L}{2}\right)$$

$$= \frac{WL^4}{96} - \frac{WL^4}{384} - \frac{WL^4}{48} = -\frac{5WL^4}{384}$$

$$\therefore y_c = -\frac{5WL^4}{384EI}$$

(Negative sign shows that deflection is downwards)

$\therefore$  Maximum deflection,

$$y_c = \frac{5WL^4}{384EI}$$

**Example.3.2.1.** A cantilever of length 3 m is carrying a point load of 25 kN at the free end. If the moment of inertia of the beam =  $10^8 \text{ mm}^4$  and value of  $E = 2.1 \times 10^5 \text{ N/mm}^2$ , find (i) slope of the cantilever at the free end and (ii) deflection at the free end.

**Sol. Given:**

Length,  $L = 3\text{m} = 3000\text{ mm}$

Point load,  $W = 25\text{kN} = 25000\text{ N}$

M.O.I.,  $I = 10^8\text{ mm}^4$

Value of  $E = 2.1 \times 10^5\text{ N/mm}^2$

(i) Slope at the free end is given by equation.

$$\therefore \theta_B = \frac{WL^2}{2EI} = \frac{25000 \times 3000^2}{2 \times 2.1 \times 10^5 \times 10^8} = \mathbf{0.005357\text{ rad.}} \quad \text{Ans.}$$

(ii) Deflection at the free end is given by equation

$$y_B = \frac{WL^3}{3EI} = \frac{25000 \times 3000^3}{3 \times 2.1 \times 10^5 \times 10^8} = \mathbf{10.71\text{ mm.}} \quad \text{Ans.}$$

**Example.3.2.2** A cantilever of length 3 m is carrying a point load of 50 kN at a distance of 2 m from the fixed end. If  $I = 10^5\text{ mm}^4$  and value of  $E = 2 \times 10^5\text{ N/mm}^2$ , find (i) slope at the free end and (ii) deflection at the free end.

**Sol. Given:**

Length,  $L = 3\text{m} = 3000\text{ mm}$

Point load,  $W = 50\text{ kN} = 50000\text{ N}$

Distance between the load and the fixed end,

$$a = 2\text{ m} = 2000\text{ mm}$$

M.O.I.,  $I = 10^8\text{ mm}^4$  Value of  $E = 2 \times 10^5\text{ N/mm}^2$

(i) Slope at the free end is given by equation as

$$\therefore \theta_B = \frac{Wa^2}{2EI} = \frac{50000 \times 2000^2}{2 \times 2 \times 10^5 \times 10^8} = \mathbf{0.005\text{ rad.}} \quad \text{Ans.}$$

(ii) Deflection at the free end is given by equation as

$$\begin{aligned} y_B &= \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI} (L - a) \\ &= \frac{50000 \times 2000^3}{3 \times 2 \times 10^5 \times 10^8} + \frac{50000 \times 2000^2}{2 \times 2 \times 10^5 \times 10^8} (3000 - 2000) \\ &= 6.67 + 5.0 = \mathbf{11.67\text{ mm.}} \quad \text{Ans.} \end{aligned}$$

**Example 3.2.3.** A cantilever of length 2m carries a uniformly distributed load of 2.5kN/m run for a length of 1.25m from the fixed end and a point load of 1 kN at the free end. Find the deflection at the free end if the section is rectangular 12 cm wide and 24 cm deep and  $E = 1 \times 10^4\text{ N/mm}^2$

**Given Data:**

Length,  $L = 2\text{m} = 2000\text{mm}$

u.d.l,  $w = 2.5 \text{ kN/m} = 2.5 \times \frac{1000}{1000} \text{ N/mm} = 2.5 \text{ N/mm}$  run for a  
 'length of 'a'  
 $= 1.25\text{m} = 1250\text{mm}$  from the fixed end.

Point load at the free end

$$W = 1 \text{ kN} = 1000\text{N}$$

Width,  $b = 12\text{cm} = 120\text{mm}$

Depth,  $d = 24\text{cm} = 240\text{mm}$

$E = 1 \times 10^4 \text{ N/mm}^2$  **To Find:** The deflection at the free end **Solution:**

Moment of inertia of the rectangular section

$$I = \frac{bd^3}{12} = \frac{120 \times 240^3}{12} = 13824 \times 10^4 \text{ mm}^4$$

Downward deflection at the free end due to point load

$$y^1 = \frac{WL^3}{3EI} = \frac{1000 \times 2000^3}{3 \times 10^4 \times 13824 \times 10^4} = 1.929 \text{ mm.}$$

Downward deflection at the free end due to uniformly distributed load run over  $1.25\text{m}$  from the fixed end.

$$\begin{aligned} y^2 &= \frac{Wa^4}{8EI} + \frac{Wa^3}{6EI}(L-a) \\ &= \frac{2.5 \times 1250^4}{8 \times 10^4 \times 13824 \times 10^4} + \frac{2.5 \times 1250^3}{6 \times 10^4 \times 13824 \times 10^4} (2000-1250) \\ &= 0.9934 \text{ mm} \end{aligned}$$

$\therefore$  Total deflection at the free end due to point load and u.d.l

$$= y_1 + y_2 = 1.929 + 0.9934 = \mathbf{2.9224 \text{ mm}}$$

**Example.3.2.4.** A cantilever of length  $2\text{m}$  carries a uniformly distributed load  $2 \text{ kN/m}$  over a length of  $1\text{m}$  from the free end, and a point load of  $1 \text{ kN}$  at the free end. Find the slope and deflection at the free end if  $E = 2.1 \times 10^5 \text{ N/mm}^2$  and  $I = 6.667 \times 10^7 \text{ mm}^4$ .

**Given Data**

Length,  $L = 2\text{m} = 2000\text{mm}$

u.d.l,  $w = 2 \text{ kN/m} = 2 \times \frac{1000}{1000} \text{ N/mm} = 2 \text{ N/mm}$  run for a length of  $1\text{m} =$

1000mm from the free end

Point load at the free end

$$W = 1 \text{ kN} = 1000\text{N}$$

$$E = 2.1 \times 10^5 \text{ N/mm}^2$$

$$I = 6.667 \times 10^7 \text{ mm}^4$$

**To Find:** The slope and deflection at the free end Solution:

**Slope at the free end.**

The slope at the free end due to a point load

$$\begin{aligned} \theta_1 &= \frac{WL^2}{2EI} \\ &= \frac{1000 \times 2000^2}{2 \times 2.1 \times 10^5 \times 6.667 \times 10^7} \\ &= 0.0001428 \text{ radians.} \end{aligned}$$

The slope at the free end due to u.d.l of 2 kN/m over a length of 1m from the free end.

$$\begin{aligned} \theta_2 &= \frac{WL^3}{6EI} - \frac{W(L-a)^3}{6EI} \\ &= \frac{2 \times 2000^3}{6 \times 2.1 \times 10^5 \times 6.667 \times 10^7} - \frac{2 \times (2000-1000)^3}{6 \times 2.1 \times 10^5 \times 6.667 \times 10^7} \\ &= 0.0001666 \text{ radians.} \end{aligned}$$

$$\therefore \text{Total slope at the free end} = \theta_1 + \theta_2$$

$$= 0.0001428 + 0.0001666$$

$$= \mathbf{0.0003094 \text{ radians}}$$

**Deflection at the free end.**

The Deflection at the free end due to a point load

$$\begin{aligned} y^1 &= \frac{WL^3}{3EI} \\ &= \frac{1000 \times 2000^3}{3 \times 2.1 \times 10^5 \times 6.667 \times 10^7} \\ &= 0.1904\text{mm.} \end{aligned}$$

The Deflection at the free end due to u.d.l of 2 kN/m over a length of 1m from the free end.

$$\begin{aligned} y^2 &= \frac{WL^4}{8EI} - \left[ \frac{w(L-a)^4}{8EI} + \frac{w(L-a)^3}{6EI} \times a \right] \\ &= \frac{2 \times 2000^4}{8 \times 2.1 \times 10^5 \times 6.667 \times 10^7} - \frac{2 \times (2000-1000)^4}{8 \times 2.1 \times 10^5 \times 6.667 \times 10^7} - \frac{2 \times (2000-1000)^3 \times 1000}{6 \times 2.1 \times 10^5 \times 6.667 \times 10^7} \end{aligned}$$



$$= 0.244\text{mm}$$

$$\therefore \text{Total deflection at the free end} = y_1 + y_2 \\ = 0.1904 + 0.244\text{mm} = \mathbf{0.4344\text{mm}}$$

**Example.3.2.5.** A beam 6m long, simply supported at its ends, is carrying a point load of 50 kN at its centre. The moment of inertia of the beam is equal to  $78 \times 10^6 \text{ mm}^4$ . If  $E$  for the material of the beam  $= 2.1 \times 10^5 \text{ N/mm}^2$ , Calculate the slope at the supports and the deflection at the centre of the beam.

**Given Data:**

Length,  $L = 6\text{m} = 6000\text{mm}$

Point load,  $W = 50\text{kN} = 50000\text{N}$

M.O.I  $I = 78 \times 10^6 \text{ mm}^4$

Value of  $E = 2.1 \times 10^5 \text{ N/mm}^2$  **To Find:**

The maximum slope and Deflection.

**Solution:**

**Maximum slope at supports**

$$\theta_B = \theta_A = -\frac{WL^2}{16EI} \\ = \frac{WL^2}{16EI} \\ = \frac{50000 \times 6000^2}{16 \times 2.1 \times 10^5 \times 78 \times 10^6} \\ = \mathbf{0.06868 \text{ radians.}}$$

**Maximum deflection at centre**

$$y_c = \frac{WL^3}{48EI} = \frac{50000 \times 6000^3}{48 \times 2.1 \times 10^5 \times 78 \times 10^6} = \mathbf{13.736\text{mm.}}$$

**Example.3.2.6.** A beam 4m long, simply supported at its ends, carries a point load  $W$  at its centre. If the slope at the ends of the beam is not to exceed 1 degree. Find the deflection at the centre of the beam **Given Data:**

Length,  $L = 4\text{m} = 4000\text{mm}$  Point load at centre,  $= W$

$$\text{Slope at supports, } \theta_B = \theta_A = 1^\circ = \frac{1 \times \pi}{180} = 0.01745 \text{ radians.}$$

We know that slope at supports,  $\theta_A = \frac{WL^2}{16EI} = 0.01745 \text{ radians.}$  Maximum deflection at centre

$$y_c = \frac{WL^3}{48EI} = \frac{WL^2}{16EI} \times \frac{L}{3}$$

$$= 0.01745 \times \frac{4000}{3}$$

$$= 23.26\text{mm.}$$

**Example.3.2.7.** A beam of uniform rectangular section 200mm wide and 300mm deep is simply supported at its ends. It carries a uniformly distributed load of 9kN/m run over the entire span of 5m. If the value of E for the material is  $1 \times 10^4 \text{N/mm}^2$ , Find the slope at the supports and maximum deflection.

**Solution.**

Moment of inertia of the rectangular section

$$I = \frac{bd^3}{12} = \frac{200 \times 300^3}{12} = 4.5 \times 10^8 \text{mm}^4$$

**Maximum slope at supports,**

$$\theta_B = \theta_A = \frac{WL^3}{24EI} = \frac{9 \times 5000^3}{24 \times 1 \times 10^4 \times 4.5 \times 10^8} = 0.0104 \text{ radians}$$

**Maximum Deflection at centre**

$$y_c = \frac{5WL^4}{384EI} = \frac{5 \times 9 \times 5000^4}{384 \times 1 \times 10^4 \times 4.5 \times 10^8} = 16.27\text{mm.}$$