2.8 ROUTH HURWITZ CRITERION

Consider a closed-loop transfer function

$$H(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} = \frac{B(s)}{A(s)}$$

where the a_i's and b_i's are real constants and m ≤ n. An alternative to factoring the denominator polynomial, Routh's stability criterion, determines the number of closed-loop poles in the right-half s-plane.

Algorithm for applying Routh's stability criterion

The algorithm described below, like the stability criterion, requires the order of A(s) to be finite.

1. Factor out any roots at the origin to obtain the polynomial, and multiply by −1 if necessary, to obtain

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

$$where, a_0 \neq 0 \; and \; a_n > 0$$

2. If the order of the resulting polynomial is at least two and any coefficient a_i is zero or negative, the polynomial has at least one root with nonnegative real part. To obtain the precise number of roots with nonnegative real part, proceed as follows. Arrange the coefficients of the polynomial, and values subsequently calculated from them as shown below:

s^n	a_0	a_2	a_4	a_6	⋯
s^{n-1}	a_1	a_3	a_5	a_7	⋯
s^{n-2}	b_1	b_2	b_3	b_4	⋯
s^{n-3}	c_1	c_2	c_3	c_4	⋯
s^{n-4}	d_1	d_2	d_3	d_4	⋯
⋮	⋮	⋮			
s^2	e_1	e_2			
s^1	f_1				
s^0	g_0				

The array is generated until all subsequent coefficients are zero. Similarly, cross multiply the coefficients of the two previous rows to obtain the c_i, d_i, etc. Until the nth row of the array has been completed. Missing coefficients are replaced by zeros. The resulting array is called the Routh array. The powers of s are not considered to be part of the array. We can think of them as labels. The column beginning with a0 is considered to be the first column of the array. The Routh array is seen to be triangular. It can be shown that multiplying a row by a positive number to simplify the calculation of the next row does not affect the outcome of the application of the Routh criterion. where, the coefficients b_i are,

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

$$\vdots$$

3. Count the number of sign changes in the first column of the array. It can be shown that a necessary and sufficient condition for all roots of (2) to be located in the left-half plane is that all the a_i are positive and all of the coefficients in the first column be positive.

Example: Generic Cubic Polynomial

Consider the generic cubic polynomial:

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

where all the ai are positive. The Routh array is

s^3	a_0	a_2
s^2	a_1	a_3
s^1	$\frac{a_1 a_2 - a_0 a_3}{a_0}$	
s^0	a_3	

So, the condition that all roots have negative real parts is

$$a_1 a_2 > a_0 a_3$$

Example: A Quadratic Polynomial.

Next, we consider the fourth-order polynomial:

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Here we illustrate the fact that multiplying a row by a positive constant does not change the result. One possible Routh array is given at left, and an alternative is given at right,

s^4	1	3	5
s^3	2	4	0
s^2	1	5	
s^1	-6		
s^0	5		

Also,

s^4	1	3	5
s^3	~~2~~	~~4~~	~~0~~
	1	2	0
s^2	1	5	
s^1	-3		
s^0	5		

In this example, the sign changes twice in the first column so the polynomial equation A(s) = 0 has two roots with positive real parts.

Necessity of all coefficients being positive

In stating the algorithm above, we did not justify the stated conditions. Here we show that all coefficients being positive is necessary for all roots to be located in the left half-plane. It can be shown that any polynomial ins, all of whose coefficients are real, can be factored into a product of a maximal number linear and quadratic factors also having real coefficients. Clearly a linear factor (s+a) has nonnegative real root if a is positive. For both roots of a quadratic factor (s2+bs+c) to have negative real parts both b and c must be positive. (If c is negative, the square root ofb2−4cis real and the quadratic factor can be factored into two linear factors so the number of factors was not maximal.) It is easy to see that if all coefficients of the factors are positive, those of the original polynomial must be as well. To see that the condition is not sufficient, we can refer to several examples above.

Example: Determining Acceptable Gain Values

Consider a system whose closed-loop transfer function is

$$H(s) = \frac{K}{s(s^2 + s + 1)(s + 2) + K}$$

Characteristic equation

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

Routh array is

s^4	1	3	K
s^3	3	2	
s^2	7/3	K	
s^1	(14-9K)/7		
s^0	K		

For the system to be stable, the elements of the first column of the Routh array should be positive. Based on that condition, the s^1 row yields the condition that, for stability,

$$\frac{(14 - 9K)}{7} > 0$$

$$(14 - 9K) > 0$$

$$14 > 9K$$

$$\frac{14}{9} > K$$

The s^0 row yields the condition that, for stability,

$$K > 0$$

Hence, the system is stable when the value of K lies in the range of

$$0 < K < 14/9$$

Special Case: Zero First-Column Element.

If the first term in a row is zero, but the remaining terms are not, the zero is replaced by a small, positive value of ϵ and the calculation continues as described above. Here's an example:

$$s^3 + 2s^2 + s + 2 = 0$$

Routh array is

s^3	1	1
s^2	2	2
s^1	$0 \cong \epsilon$	
s^0	2	

Special Case: Zero Row

If all the coefficients in a row are zero, a pair of roots of equal magnitude and opposite sign is indicated. These could be two real roots with equal magnitudes and opposite signs or two conjugate imaginary roots. The zero row is replaced by taking the coefficients of dP(s)/ds, where P(s), called the auxiliary polynomial, is obtained from the values in the row above the zero row. The pair of roots can be found by solving dP(s)/ds= 0. Note that the auxiliary polynomial always has even degree. It can be shown that an auxiliary polynomial of degree 2n has n pairs of roots of equal magnitude and opposite sign.