UNIT –II

COMPLEX INTEGRATION

LINE INTEGRAL AND CONTOUR INTEGRAL

If $f(z)$ is a continuous function of the complex variable $z = x + iy$ and C is any continuous curve connecting two points A and B on the z – plane then the complex line integral of $f(z)$ along C from A to B is denoted by $\int_c f(z)dz$

When C is simple closed curve, then the complex integral is also called as a contour integral and is denoted as $\oint_C f(z)dz$. The curve C is always take in the anticlockwise direction.

Note: If the direction of C is reversed (clockwise), the integral changes its sign

$$(ie) \oint_c f(z)dz = -\oint f(z)dz$$

Standard theorems:

1. **Cauchy's Integral theorem (or) Cauchy's Theorem (or) Cauchy's Fundamental Theorem**

   Statement: If $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve C then $\oint_c f(z)\,dz = 0$

2. **Extension of Cauchy's integral theorem (or) Cauchy's theorem for multiply connected Region**

   Statement: If $f(z)$ is analytic at all points inside and on a multiply connected region whose outer boundary is C and inner boundaries are $C_1, C_2, \dots, C_n$ then

$$\int_c f(z)dz = \int_c f(z)dz + \int_{C_2} f(z)dz + \cdots + \int_{c_n} f(z)dz$$

3. **Cauchy's integral formula**

   Statement: If $f(z)$ is analytic inside and on a simple closed curve C of a simply connected region R and if 'a' is any point interior to C, then

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}dz$$

(OR)

$$\int_c \frac{f(x)}{z-a}dz = 2\pi i\, f(a),$$

   the integration around C being taken in the positive direction.

4. **Cauchy's Integral formula for derivatives**

   Statement: If $f(z)$ is analytic inside and on a simple closed curve C of a simply connected Region R and if 'a' is any point interior to C, then

$$\int_c \frac{f(z)}{(z-a)^2}dz = 2\pi i f'(a)$$

$$\int_c \frac{f(z)}{(z-a)^3} dz = 2\pi i f''(a)$$

In general, $\int_c \frac{f(z)}{(z-a)^n} dz = 2\pi i f^{(n-1)}(a)$

Problems based on Cauchy's Integral Theorem

Example: 4.1 Evaluate $\int_o^{3+i} z^2 dz$ along the line joining the points (0, 0) and (3, 1)

Solution:

Given $\int_0^{3+i} z^2 dz$

Let $z = x + iy$

Here $z = 0$ corresponds to (0, 0) and $z = 3 + i$ corresponds to (3, 1)

The equation of the line joining (0, 0) and (3, 1) is

$$y = \frac{x}{3} \Rightarrow x = 3y$$

Now $z^2 dz = (x + iy)^2 (dx + idy)$

$$= [x^2 - y^2 + i2xy][dx + idy]$$

$$= [(x^2 - y^2) + i2xy][dx + idy]$$

$$= [(x^2 - y^2)dx - 2xydy] + i[2xydx + (x^2 - y^2)dy]$$

Since $x = 3y \Rightarrow dx = 3dy$

$$\therefore z^2 dz = [8y^2(3dy) - 6y^2 dy] + i[18y^2 dy + 8y^2 dy]$$

$$= 18y^2 dy + i26y^2 dy$$

$$\therefore \int_0^{3+i} z^2 dz = \int_0^{'} [18y^2 + i26y^2] dy$$

$$= \left[18 \frac{y^2}{3} + i\, 26 \frac{y^3}{3}\right]_0^{'}$$

$$= 6 + i\frac{26}{3}$$

Example: 4.2 Evaluate $\int_0^{2+i} (x^2 - iy) dz$

Solution:

Let $z = x + iy$

Here $z = 0$ corresponds to (0, 0) and $z = 2 + i$ corresponds to (2, 1)

Now $(x^2 - iy)dz = (x^2 - iy)(dx + idy)$

$$= x^2 dx + y\, dy) + i\, (x^2 dy - y\, dx)$$

Along the path $y = x^2 \Rightarrow dy = 2xdx$

$$\therefore (x^2 - iy)dz = (x^2 dx + 2x^3 dx) + i(2x^3 dx - x^2 dx)$$

$$\int_0^{2+i} (x^2 - iy)dz = \int_0^2 (x^2 + 2x^3)dx + i(2x^3 - x^2)dx$$

$$= \left[\frac{x^3}{3} + \frac{2x^4}{4}\right]_0^2 + i\left[\frac{2x^4}{4} = \frac{x^3}{3}\right]_0^2$$

$$= \left(\frac{8}{3} + \frac{16}{2}\right) + i\,\left(\frac{16}{2} - \frac{8}{3}\right)$$

$$= \frac{32}{3} + i\ \frac{16}{3}$$

Example: 4.3 Evaluate $\int_c e^{\frac{1}{z}}\, \boldsymbol{dz, where\ C\ is\ |z| = 2}$

Solution:

Let $f(z) = e^{\frac{1}{z}}$ clearly $f(z)$ is analytic inside and on C.

Hence, by Cauchy's integral theorem we get $\int_c e^{\frac{1}{z}} dz = 0$

Example: 4.4 Evaluate $\int_c \boldsymbol{z^2 e^{\frac{1}{z}} dz, where\ C\ is\ |z| = 1}$

Solution:

Given $\int_c z^2\, e^{1/z} dz$

$$= \int_c \frac{z^2}{e^{-1/z}} dz$$

$Dr = 0 \implies z = 0,$ We get $e^{-\frac{1}{0}} = e^{-\infty} = 0$

$z = 0$ lies inside $|z| = 1$.

Cauchy's Integral formula is

$$\int_c z^2\, e^{1/z} dz = 2\pi i f(0) = 0$$

Example: 4.5 Evaluate $\int_c \boldsymbol{\frac{1}{2z-3} dz\ \ where\ C\ is\ |z| = 1}$

Solution:

Given $\int_c \frac{1}{2z-3} dz$

$Dr = 0 \implies 2z - 3 = 0, \implies z = \frac{3}{2}$

Given $C\ is\ |z| = 1$

$$\Rightarrow |z| = \left|\frac{3}{2}\right| = \frac{3}{2} > 1$$

$\therefore z = \frac{3}{2}$ lies outside C

$\therefore$ By Cauchy's Integral theorem, $\int_c \frac{1}{2z-3} dz = 0$

Example: 4.6 Evaluate $\int_c \boldsymbol{\frac{dz}{z+4}\ \ where\ C\ is\ |z| = 2}$

Solution:

Given $\int_c \frac{dz}{z+4}$

$Dr = 0 \implies z + 4 = 0 \implies z = -4$

Given $C\ is\ |z| = 2$

$$\Rightarrow |z| = |-4| = 4 > 2$$

$\therefore z = -4$ lies outside C.

$\therefore$ By Cauchy's Integral Theorem, $\int_c \frac{dz}{z+4} = 0$

Example: 4.7 Evaluate $\int_c \boldsymbol{\frac{e^{2z}}{z^2+1} dz, where\ C\ is\ |z| = \frac{1}{2}}$

Solution:

Given $\int_c \frac{e^{2z}}{z^2+1} dz$

$Dr = 0 \implies z^2 + 1 = 0 \implies z = \pm i$

Given $C \ is \ |z| = \frac{1}{2}$

$$\Rightarrow |z| = |\pm i| = 1 > \frac{1}{2}$$

∴Clearly both the points $z = \pm i$ lies outside C.

∴ By Cauchy's Integral Theorem, $\int_c \frac{e^{2z}}{z^2+1} dz = 0$

Example: 4.8 Using Cauchy's integral formula Evaluate $\int_c \frac{z+1}{(z-3)(z-1)} dz, where \ C \ is \ |z| = 2$

Solution:

Given $\int_c \frac{z+1}{(z-3)(z-1)} dz$

$Dr = 0 \implies z = 3, 1$

Given $C \ is \ |z| = 2$

∴Clearly $z = 1$ lies inside C and $z = 3$ lies outside C

$$\int_c \frac{z+1}{(z-3)(z-1)} dz = \int_c \frac{(z+1)/(z-3}{(z-1)} dz$$

∴ By Cauchy's Integral Theorem

$\int_c \frac{(z+1)/(z-3)}{(z-1)} dz = 2\pi i f(1)$ Where $f(z) = \frac{z+1}{z-3} \Rightarrow f(1) = \frac{2}{-2}$

$$= 2\pi i(-1) = -2\pi i$$

Example: 4.9 Using Cauchy's integral formula, evaluate $\int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)(z-3)} dz$ where C is the circle $|z| = 4$.

Solution:

Given $\int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)(z-3)} dz$

$Dr = 0 \implies z = 2, 3$

Given $C \ is \ |z| = 4$

∴Clearly $z = 2 \ and$ 3 lies inside C.

Consider, $\frac{1}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$

$$\Rightarrow 1 = A(z-3) + B(Z = 2)$$

Put $z = -3 \Rightarrow 1 = B$

Put $z = 2 \Rightarrow -1 = A$

$\therefore \frac{1}{(z-2)(z-3)} = -\frac{1}{z-2} + \frac{1}{z-3}$

$$\int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)(z-3)} dz = -\int \frac{sin\pi z^2 + \cos\pi z^2}{z-2} dz + \int \frac{sin\pi z^2 + \cos\pi z^2}{z-3} dz$$

$= -2\pi i f(2) + 2\pi i \ f(3)$ Where $f(z) = \sin(\pi z^2) + \cos \pi z^2$

$= -2\pi i(1) + 2\pi i(-1)$ $f(2) = \sin 4\pi + \cos 4\pi = 1$

$= -4\pi i$ $\qquad\qquad$ $f(3) = \sin 9\pi + \cos 9\pi - 1 = -1$

Example: 4.10 Evaluate $\int_c \frac{z+4}{z^2+2z+5}$ Where C is the circle $(i)|z+1+i| = 2$ $(ii)|z+1-i| = 2$ (iii) $|z| = 1$

Solution:

Given $\int_c \frac{z+4}{z^2+2z+5} dz$

$Dr = 0 \Rightarrow z^2 + 2z + 5 = 0$

$\Rightarrow z = \frac{-2\pm\sqrt{4-20}}{2}$

$\Rightarrow z = -1 \pm 2i$

$\therefore \int_c \frac{z+4}{z^2+2z+5} dz = \int_c \frac{(z+4)\, dz}{[z-(-1+2i)[z-(-1-2i)]}$

(i) $|z+1+i| = 2$ *is the circle*

When $z = -1+2i, |-1+2i+1+i| = |3i| > 2$ lies outside C.

When $z = -1-2i, |-1-2i+1+i| = |-i| < 2$ lies inside C.

$\therefore$ By Cauchy's Integral formula

$$\int_c \frac{[(z+1)/(z-(-1+2i)]}{[z-(-1-2i)]} dz = 2\pi i f(-1-2i) \qquad \text{Where } f(z) = \frac{z+4}{[z-(-1+2i)]}$$

$$= 2\pi i \left[\frac{3-2i}{-4i}\right] \qquad f(-1-2i) = \frac{-1-2i+4}{-1-2i+1-2i} = \frac{3-2i}{-4i}$$

$$= \frac{\pi}{2}(2i-3)$$

(ii) $|z+1-i| = 2$ is the circle

When $z = -1+2i, |-1+2i+1-i| = |i| < 2$ lies inside C

When $z = -1-2i, |-1-2i+1-i| = |-3i| > 2$ lies outside C

$\therefore$ By Cauchy's Integral formula

$$\int_c \frac{(z+1)/[z-(-1-2i)]}{[z-(-1+2i)]} dz = 2\pi i\, f(-1+2i) \qquad \text{Where } f(z) = \frac{z+4}{z-(-1-2i)}$$

$$= 2\pi i \frac{[3+2i]}{4i} \qquad f(-1+2i) = \frac{-1+2i+4}{-1+2i+1+2i} = \frac{3+2i}{4i}$$

$$= \frac{\pi}{2}(3+2i)$$

(iii)$|z| = 1$ is the circle

When $z = -1+2i, 1-1+2i| = \sqrt{5} > 1$ lies outside C

When $z = -1-2i, 1-1-2i| = \sqrt{5} > 1$ lies outside C

$\therefore$ By Cauchy's Integral theorem

$$\int_c \frac{z+4}{z^2+2z+5} dz = 0$$

Example: 4.11 Using Cauchy's integral formula, evaluate $\int_c \frac{z+1}{z^2+2z+4} dz$ where C is the circle $|z+1+i| = 2$

Solution:

Given $\int_c \frac{z+1}{z^2+2z+4} dz$

$Dr = 0 \Rightarrow z^2 + 2z + 4 = 0$

$\Rightarrow z = \frac{-2\pm\sqrt{4-16}}{2}$

$\Rightarrow z = -1 \pm i\sqrt{3}$

$\therefore \int_c \frac{z+1}{z^2+2z+4} dz = \int_c \frac{(z+1\ )dz}{[z-(-1+i\sqrt{3})][z-(-1-i\sqrt{3}]}$

Given $C\ is\ |z + 1 + i| = 2$

When $z = -1 - i\sqrt{3}, \left|-1 - i\sqrt{3} + 1 + i\right| = |(1 - \sqrt{3}i)| < 2$ lies inside C.

When $z = -1 + i\sqrt{3}, \left|-1 + i\sqrt{3} + 1 + i\right| = |i + \sqrt{3}i| > 2$ lies outside C.

∴ By Cauchy's Integral Formula

$\int_c \frac{(z+1)/[z-(-1+i\sqrt{3})]}{[z-(-1-i\sqrt{3})]} dz = 2\pi i\, f(-1 - i\sqrt{3})$ Where $f(z) = \frac{z+1}{z-(-1+i\sqrt{3})}$

$= 2\pi i \left(\frac{1}{2}\right) = \pi i$ $f(-1 - i\sqrt{3}) = \frac{-1-i\sqrt{3}+1}{-1-i\sqrt{3}+1-i\sqrt{3}} = \frac{\sqrt{3}i}{-2i\sqrt{3}} = \frac{1}{2}$

$\therefore \int_c \frac{z+1}{z^2+2z+4} dz = \pi i$

Example: 4.12 Evaluate $\int_c \frac{z^2+1}{z^2-1} dz$ where C is the circle $(i)|z - 1| = 1\ (ii)|z + 1| = 1(iii)|z - i| = 1$

Solution:

Given $\int_c \frac{z^2+1}{z^2-1} dz = \int_c \frac{z^2+1}{(z+1)(z-1)} dz$

$Dr = 0 \Longrightarrow z = 1, -1$

(i) $(z - 1) = 1$ is the circle

When $z = 1$, $|1 - 1| = 0 < 1$ lies inside C

When $z = -1, |-1 - 1| = 2 > 1$ lies outside C

∴ By Cauchy's Integral formula

$\int_c \frac{z^2+1}{(z+1)(z-1)} dz = \int_c \frac{(z^2+1)/z+1}{(z-1)} dz$

$= 2\pi i f(1)$ where $f(z) = \frac{z^2+1}{z+1} \Rightarrow f(1) = 1$

$= 2\pi i(1)$

$= 2\pi i$

(ii)$|z + 1| = 1$ is the circle

When $z = 1, |1 + 1| = 2 > 1$ lies outside C

When $z = -1, |-1 + 1| = 0 < 1$ lies inside C

∴ By Cauchy's Integral formula

$\int_c \frac{(z^2+1)/(z-1)}{z+1} dz = 2\pi i\, f(-1)$ where $f(z) = \frac{z^2+1}{z-1} \Rightarrow f(-1) = -1$

$= 2\pi i(-1) = -2\pi i$

(iii) $|z - i| = 1$ is the circle

When $z = 1, |1 - i| = \sqrt{2} > 1$ lies outside C

When $z=-1, |-1-i|=\sqrt{2}>1$ lies outside C

∴ By Cauchy's Integral Formula

$$\int_c \frac{(z^2+1)}{(z+1)(z-1)} dz = 0$$

Problems based on Cauchy's Integral Formula for derivatives

Example: 4.13 If $f(a) = \int_c \frac{3z^2+7z+1}{z-a} dz$ where C is the circle $x^2+y^2=4$ find the values of $f(3), f(1), f'(1-i)$ and $f''(1-i)$

Solution:

Given $f(a) = \int_c \frac{3z^2+7z+1}{z-a} dz$

To find: $f(3) = \int_c \frac{3z^2+7z+1}{z-3} dz$

$Dr = 0 \Longrightarrow z = 3$

Hence $z=3$ lies outside the circle $x^2+y^2=4$

By Cauchy's Integral theorem

$$\int_c \frac{3x^2+7z+1}{z-3} dz = 0$$

To find: $f(1) = \int_c \frac{3z^2+7z+1}{z-1} dz$

$Dr = 0 \Longrightarrow z = 1$

Clearly $z=1$ lies inside the circle $x^2+y^2=4$

∴ By Cauchy's Integral formula

$$\int_c \frac{3z^2+7z+1}{z-1} dz = 2\pi i\, f(1) \qquad \text{Where } f(z) = 3z^2+7z+1 \Rightarrow f(1) = 11$$

$$= 2\pi i(11)$$

$$= 22\pi i$$

To find: $f'(1-i) = \int_c \frac{3z^2+7z+1}{z-(1-i)} dz$

$Dr = 0 \Longrightarrow z = 1-i$

and the point $z=1-i$ lies inside the circle $x^2+y^2=4$

∴ By Cauchy's Integral formula

$$f'(1-i) = 2\pi i \varphi'(1-i) \qquad \text{Where } \varphi(z) = 3z^2+7z+1$$

$$= 2\pi i[6(1-i)+7] \qquad \Rightarrow \varphi'(z) = 6z+7$$

$$= 2\pi i[13-6i] \qquad \Rightarrow \varphi'(1-i) = 6(1-i)+7$$

$$= 2\pi i[13-6i]$$

To find: $f''(1-i) = \int_c \frac{3z^2+7z+1}{z-(1-i)} dz$

Cleary and the point $z=1-i$ lies inside the circle $x^2+y^2=4$

∴ By Cauchy's Integral formula

$$f'(1-i) = 2\pi i \varphi''(1-i) \qquad \text{Where } \varphi(z) = 3z^2+7z+1$$

$= 2\pi i[6]$ $\qquad \varphi''(z) = 6z + 7 \Rightarrow \varphi''(z) = 6$

$= 12\pi i$

Example: 4.14 Using Cauchy's Integral formula evaluate $\int_c \frac{zdz}{(z-1)(z-2)^2}$ where C is the circle $|z-2| = \frac{1}{2}$

Solution:

Given $\int_c \frac{zdz}{(z-1)(z-2)^2}$

$Dr = 0 \Longrightarrow z = 1$ is a pole of order 1, $z = 2$ is a pole of order 2.

Given C is $|z-2| = \frac{1}{2}$

When $z = 1, |1-2| = 1 > \frac{1}{2}$ lies outside C.

When $z = 2, |2-2| = 0 < \frac{1}{2}$ lies inside C.

∴ By Cauchy's Integral formula

$$\int_c \frac{z/z-1}{(z-2)^2} dz = 2\pi i\, f'(2) \qquad \text{Where } f(z) = \frac{z}{z-1}$$

$$= 2\pi i(-1) \qquad f'(z) = \frac{(z-1)1 - z(1)}{(z-1)^2} \Rightarrow f'(2) = -1$$

$$= -2\pi i$$

Example: 4.15 Evaluate $\int_c \frac{sin^2 z}{\left(z-\frac{\pi}{6}\right)^3} dz$ where C is the circle $|z| = 1$

Solution:

Given $\int_c \frac{sin^2 z}{(z-\frac{\pi}{6})^3} dz$

$Dr = 0 \Rightarrow z = \frac{\pi}{6}$ is a pole of order 3.

Give C is $|z| = 1$.

Clearly $z = \frac{\pi}{6}$ lies inside the circle $|z| = 1$

∴ By Cauchy's Integral formula

$$\int_c \frac{sin^2 z}{(z-\frac{\pi}{6})^3} dz = \frac{2\pi i}{2!} f''(\pi/6) \qquad \text{Where } f(z) = sin^2 z$$

$$= \frac{2\pi i}{2!}(1) \qquad f'(z) = 2\sin z \cos z = \sin 2z$$

$$= \pi i \qquad f''(z) = \cos 2z(2) \Rightarrow f''\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{2\pi}{6}\right)$$

$$= 2\cos\frac{\pi}{3} = 2\left(\frac{1}{2}\right) = 1$$

Example: 4.16 Evaluate $\int_c \frac{z}{(z-1)^3} dz$ where C is the circle $|z| = 2$, using Cauchy's Integral formula

Solution:

Given $\int_c \frac{z}{(z-1)^3} dz$

$Dr = 0 \Rightarrow z = 1$ is a pole of order 3.

Given C is $|z| = 2$.

Clearly $z = 1$ lies inside the circle C

∴ By Cauchy's Integral formula

$$\int_c \frac{sin^2 z}{(z-1)^3} dz = \frac{2\pi i}{2!} f''(1) \qquad \text{Where } f(z) = z \Rightarrow f'(z) = 1$$

$$= \frac{2\pi i}{2!}(0) \qquad \Rightarrow f''(z) = 0 \Rightarrow f''(1) = 0$$

$$= 0$$

Example: 4.17 Evaluate $\int_c \frac{z^2}{(2z-1)^2} dz$ **where C is the circle** $|z| = 1$

Solution:

Given $\int_c \frac{z^2}{(2z-1)^2} dz$

$Dr = 0 \Rightarrow 2z = 0 \Rightarrow z = \frac{1}{2}$ is a pole of order 2.

Given C is $|z| = 1$.

Clearly $z = \frac{1}{2}$ lies inside the circle C

∴ By Cauchy's Integral formula

$$\int_c \frac{z^2}{2^2(z-\frac{1}{2})^2} dz = \frac{1}{4}\int_c \frac{z^2}{\left(z-\frac{1}{2}\right)^2} dz \qquad \text{Where } f(z) = z^2 \Rightarrow f'(z) = 2z$$

$$= \frac{1}{4}\left(2\pi i f'\left(\frac{1}{2}\right)\right) \qquad \Rightarrow f'\left(\frac{1}{2}\right) = 1$$

$$= \frac{1}{2}\pi i(1)$$

$$= \frac{\pi i}{2}$$

Exercise: 4.1

Evaluate the following using Cauchy's Integral formula

1. $\int_c \frac{z^2}{z^2+9} dz$ where C is $|z-1| = \frac{3}{2}$ **Ans:** 0
2. $\int_c \frac{7z-1}{z^2-3z-4}$ where C is the ellipse $x^2 + 4y^2 = 4$ **Ans:** $\frac{16\pi i}{5}$
3. $\int_c \frac{z^3-z}{(z+2)^3} dz$ where C is $|z| = 3$ **Ans:** $12\pi i$
4. $\int_c \frac{3z-1}{z^2-z} dz$ where C is $|z| = \frac{1}{2}$ **Ans:** $2\pi i$
5. $\int_c \frac{12z-7}{(2z+3)(z-1)^3} dz$ where C is the circle $x^2 + 4y^2 = 4$ **Ans:** 0
6. $\int_c \frac{e^{2z}}{(z+1)(z-2} dz$ where C $is|z| = 3$ **Ans:** $2\pi i(e^4 - e^2)$
7. $\int_c \frac{z+1}{z^4-4z^3+4z^2} dz$ where C $is|z-2-i| = 2$ **Ans:** πi
8. $\int_c \frac{z}{z^4-4z^3+4z^2} dz$ where C $is|z-2\} = \frac{1}{2}$ **Ans:** $4\pi i$
9. $\int_c \frac{z}{(z-2)(z-3)^2} dz$ where C is $|z-3| = \frac{1}{2}$ **Ans:** $-4\pi i$

10. If $f(a) = \int_c \frac{4z^2+z+5}{z-a} dz$ where C is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ find the values of $f(1), f(i), f'(1-i)$ and $f''(1+i)$ **Ans:** $20\pi i, 2\pi i(1+i), 2\pi i\ (9-8i), 8\pi i$