2.3 Approximation of derivatives, Impulse invariance method, Bilinear transformation

Design a chebyshev filter for the following specification using bilinear transformation.

$$0.8 \le \left|H\left(e^{j\omega}\right)\right| \le 1 \qquad 0 \le \omega \le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.2 \qquad 0.6\pi \le \omega \le \pi.$$

Solution:

Given data:

Pass band attenuation $\alpha_P = 0.8$ *; Pass band frequency* $\omega_P = 0.2\pi$ *;*

Stop band attenuation $\alpha_S = 0.2$ *; Stops band frequency* $\omega_S = 0.6\pi$ *;*

Step 1: Specifying the pass band and stop band attenuation in dB.

Pass band attenuation $\alpha_P = -20\log\delta 1 = -20\log(0.8) = 1.938dB$

Stop band attenuation $\alpha_S = -20\log\delta 2 = -20\log(0.2) = 13.9794dB$

Step2. Choose *T* and determine the analog frequencies (i.e) Prewarp band edge frequency

$$\Omega_P = \frac{2}{T}\tan\left(\frac{\omega_P}{2}\right) = 2\tan\left(\frac{0.2\pi}{2}\right) = 0.649dB$$

$$\Omega_S = \frac{2}{T}\tan\left(\frac{\omega_S}{2}\right) = 2\tan\left(\frac{0.6\pi}{2}\right) = 2.75dB$$

Step3. To find order of the filter

$$N \ge \frac{Cosh^{-1}\sqrt{\dfrac{10^{0.1\alpha_S}-1}{10^{0.1\alpha_P}-1}}}{Cosh^{-1}\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

$$\ge \frac{Cosh^{-1}\sqrt{\dfrac{10^{0.1*13.97}-1}{10^{0.1*1.938}-1}}}{Cosh^{-1}\left(\dfrac{2.75}{0.649}\right)}$$

$$\ge \frac{Cosh^{-1}\sqrt{\dfrac{23.945}{0.562}}}{Cosh^{-1}\left(\dfrac{2.75}{0.649}\right)}$$

$$\ge \frac{Cosh^{-1}(6.5273)}{Cosh^{-1}(4.2372)}$$

$$\ge \frac{2.5632}{2.1228}$$

$$\ge 1.207$$

Rounding the next higher integer value N=2

Step4. The poles of chebyshev filter can be determined by

$$S_k = a\cos\varphi_k + jb\sin\varphi_k, k = 0,1,\cdots,N$$

Where,

$$\phi_k = \left[\frac{(2k+N-1)\pi}{2N}\right] \quad \text{And calculate a, b,} \varepsilon, \ \mu$$

$$\varepsilon = \sqrt{10^{0.1\alpha p} - 1},$$

$$= \sqrt{10^{0.1*1.938} - 1}$$

$$\varepsilon = 0.75$$

$$\mu = \varepsilon^{-1} + \left[\sqrt{1+\varepsilon^{-2}}\right]$$

$$= (0.75)^{-1} + \left[\sqrt{1+(0.75)^{-2}}\right]$$

$$\mu = 3$$

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right]$$

$$= 0.649\left[\frac{(3)^{\frac{1}{2}} - (3)^{-\frac{1}{2}}}{2}\right]$$

$$a = 0.375$$

$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$$

$$= 0.649\left[\frac{(3)^{\frac{1}{2}} - (3)^{-\frac{1}{2}}}{2}\right]$$

$$b = 0.75$$

$$\phi_k = \left[\frac{(2k+N-1)\pi}{2N}\right]; \quad k = 1,2$$

$$\phi_1 = \left[\frac{(2(1)+2-1)\pi}{2*2}\right] = \frac{3\pi}{4} = 135^0$$

$$\phi_2 = \left[\frac{(2(2)+2-1)\pi}{2*2}\right] = \frac{5\pi}{4} = 225^0$$

$$S_K = a\cos\varphi_k + jb\sin\varphi_k, \quad k = 1,2$$

$$for\ k = 1,$$

$$S_1 = 0.375\cos\varphi_1 + j(0.75)\sin\varphi_1$$

$$= 0.375cos135^0 + j(0.75)\sin 135^0$$

$$S_1 = -0.265 + j0.53$$

$$for\ k = 2,$$

$$S_1 = 0.375\cos\varphi_2 + j(0.75)\sin\varphi_2$$

$$= 0.375cos225^0 + j(0.75)\sin 225^0$$

$$S_1 = -0.265 - j0.53$$

Step.5 Find the denominator polynomial of the transfer function using above poles.

$$H(s) = \{S + 0.265 - j0.53\}\{S + 0.265 - j0.53\}$$
$$= \left\{(S + 0.265)^2 - (j0.53)^2\right\}$$
$$= (S + 0.265)^2 + (0.53)^2$$
$$= S^2 + 0.5306s + 0.3516$$

Step 6 : The numerator of the transfer function depends on the value of N.

- If N is Even substitute s=0 in the denominator polynomial and divide the result by $\sqrt{1+\varepsilon^2}$ Find the value. This value is equal to numerator

$$= \frac{0.3516}{\sqrt{1+\varepsilon^2}} = \frac{0.3516}{\sqrt{1+(0.75)^2}}$$

$$H(s) = 0.28$$

Step 7: The Transfer function is

$$H(s) = \frac{NM}{DM}$$

$$H(s) = \frac{0.28}{s^2 + 0.5306s + 0.3516}$$

Step 8: Apply bilinear transformation with to obtain the digital filter

$$H(z) = H(s)\Big|s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = \frac{0.28}{s^2 + 0.5306s + 0.3516}\Bigg|s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$= \frac{0.28}{s^2 + 0.5306s + 0.3516}\Bigg|s = 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$= \frac{0.28}{\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)^2 + 0.5306\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) + 0.3516}$$

$$H(z) = \frac{0.28\left(1+z^{-1}\right)^2}{1 - 1.348z^{-1} + 0.608z^{-2}}$$

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Design a chebyshev filter for the following specification using impulse invariance method.

$$0.8 \le \left|H\left(e^{j\omega}\right)\right| \le 1 \qquad 0 \le \omega \le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.2 \qquad 0.6\pi \le \omega \le \pi. [May/June - 2016]$$

Solution:

Given data:

Pass band attenuation $\alpha_P = 0.8$ *; Pass band frequency* $\omega_P = 0.2\pi$ *;*

Stop band attenuation $\alpha_S = 0.2$ *; Stops band frequency* $\omega_S = 0.6\pi$ *;*

Step 1: Specifying the pass band and stop band attenuation in dB.

$Pass\ band\ attenuation\ \alpha_P = -20\log\delta 1 = -20\log(0.8) = 1.938dB$

$Stop\ band\ attenuation\ \alpha_S = -20\log\delta 2 = -20\log(0.2) = 13.9794dB$

Step2. Choose *T* and determine the analog frequencies (i.e) Prewarp band edge frequency

$$\Omega_P = \frac{\omega_P}{T} = 0.2\pi\ Rad / Sec$$

$$\Omega_S = \frac{\omega_S}{T} = 0.6\pi\ Rad / Sec$$

Step3. To find order of the filter

$$N \geq \frac{Cosh^{-1}\sqrt{\dfrac{10^{0.1\alpha_S}-1}{10^{0.1\alpha_P}-1}}}{Cosh^{-1}\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

$$\geq \frac{Cosh^{-1}\sqrt{\dfrac{10^{0.1*13.97}-1}{10^{0.1*1.938}-1}}}{Cosh^{-1}\left(\dfrac{0.6\pi}{0.2\pi}\right)}$$

$$\geq \frac{Cosh^{-1}\sqrt{\dfrac{23.945}{0.562}}}{Cosh^{-1}(3)}$$

$$\geq \frac{Cosh^{-1}(6.5273)}{Cosh^{-1}(3)}$$

$$\geq \frac{2.5632}{1.7627}$$

$$\geq 1.454$$

Rounding the next higher integer value N=2

Step4. The poles of chebyshev filter can be determined by

$$S_k = a\cos\varphi_k + jb\sin\varphi_k, k = 0,1,\cdots,N$$

Where,

$$\phi_k = \left[\frac{(2k+N-1)\pi}{2N}\right]$$ And calculate a, b, ε, μ

$$\varepsilon = \sqrt{10^{0.1\alpha p}-1},$$

$$= \sqrt{10^{0.1*1.938}-1}$$

$$\varepsilon = 0.75$$

$$\mu = \varepsilon^{-1} + \left[\sqrt{1+\varepsilon^{-2}}\right]$$

$$= (0.75)^{-1} + \left[\sqrt{1+(0.75)^{-2}}\right]$$

$$\mu = 3$$

$$a=\Omega_p\left[\frac{\mu^{1/N}-\mu^{-1/N}}{2}\right]$$
$$=0.2\pi\left[\frac{(3)^{\frac{1}{2}}-(3)^{-\frac{1}{2}}}{2}\right]$$
$$a=0.362$$
$$b=\Omega_p\left[\frac{\mu^{1/N}+\mu^{-1/N}}{2}\right]$$
$$=0.2\pi\left[\frac{(3)^{\frac{1}{2}}-(3)^{-\frac{1}{2}}}{2}\right]$$
$$b=0.7255$$
$$\phi_k=\left[\frac{(2k+N-1)\pi}{2N}\right]\ ;\ \ k=1,2$$
$$\phi_1=\left[\frac{(2(1)+2-1)\pi}{2*2}\right]=\frac{3\pi}{4}=135^0$$
$$\phi_2=\left[\frac{(2(2)+2-1)\pi}{2*2}\right]=\frac{5\pi}{4}=225^0$$
$$S_K=a\cos\varphi_k+jb\sin\varphi_k,\ \ k=1,2$$
$$for\ k=1,$$
$$S_1=0.362\cos\varphi_1+j(0.7255)\sin\varphi_1$$
$$=0.362cos135^0+j(0.7255)\sin 135^0$$
$$S_1=-0.256+j0.513$$
$$for\ k=2,$$
$$S_1=0.362\cos\varphi_2+j(0.7255)\sin\varphi_2$$
$$=0.362cos225^0+j(0.7255)\sin 225^0$$
$$S_1=-0.256-j0.513$$

Step.5 Find the denominator polynomial of the transfer function using above poles.

$$H(s)=\{S+0.256-j0.513\}\{S+0.256-j0.513\}$$
$$=\left\{(S+0.256)^2-(j0.513)^2\right\}$$
$$=(S+0.256)^2+(0.513)^2$$
$$=S^2+0.513s+0.33$$

Step 6 : The numerator of the transfer function depends on the value of N.

- If N is Even substitute s=0 in the denominator polynomial and divide the result by $\sqrt{1+\varepsilon^2}$ Find the value. This value is equal to numerator

$$=\frac{0.33}{\sqrt{1+\varepsilon^2}}=\frac{0.33}{\sqrt{1+(0.75)^2}}$$
$$H(s)=0.264$$

Step 7: The Transfer function is

$$H(s)=\frac{NM}{DM}$$

$$H(s)=\frac{0.264}{s^2+0.513s+0.33}$$

Step 8: Using partial fraction expansion, expand H(s) into

$$H(s)=\sum_{k=1}^{2}\frac{A_k}{s-p_k}=\frac{A_1}{s-p_1}+\frac{A_2}{s-p_2}$$

$$\frac{0.264}{s^2+0.513s+0.33}=\frac{A_1}{s-(-0.256+j0.514)}+\frac{A_2}{s-(-0.256-j0.514)}$$

$$=\frac{0.257j}{s-(-0.256+j0.514)}-\frac{0.257j}{s-(-0.256-j0.514)}$$

Step 9: Now transform analog poles $\{P_k\}$ into digital poles $\{e^{p_kT}\}$ to obtain the digital filter

$$H(z)=\sum_{k=1}^{N}\frac{A_k}{1-e^{p_kT}z^{-1}}$$

$$=\sum_{k=1}^{2}\frac{A_k}{1-e^{p_kT}z^{-1}}$$

$$=\frac{0.257j}{s-e^{-0.256T}e^{j0.513T}z^{-1}}-\frac{0.257j}{s-e^{-0.256T}e^{-j0.513T}z^{-1}}$$

$$H(z)=\frac{0.1954z^{-1}}{1-1.3483z^{-1}+0.5987z^{-2}}$$

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H.W: Challenge 1: Design a chebyshev filter to meet the constraints by using bilinear transformation and assume sampling period T=1 sec.

$$\frac{1}{\sqrt{2}}\le\left|H\left(e^{j\omega}\right)\right|\le 1 \qquad 0\le\omega\le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right|\le 0.1 \qquad 0.5\pi\le\omega\le\pi.$$

Solution:

Ans: $H(z)=\dfrac{0.0413\left(1+z^{-1}\right)^2}{1-1.44z^{-1}+0.675z^{-2}}$

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============ ***H.W: Convert the following analog filter with transfer function*** $H_a(s)=\dfrac{S+0.2}{(s+0.2)^2+16}$ ***using bilinear transformation. Ans:***

$$H(z)=\frac{0.105+0.0192z^{-1}-0.0864z^{-2}}{1+1.155z^{-1}+0.9232z^{-2}}$$

Find the order and poles of a low pass Butterworth filter that has 3dB bandwidth of 500Hz and attenuation of 40dB at 1kHz.

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<u>Filter design using frequency translation (HPF, BPF, BRF):</u>

A digital filter can be converted into a digital high pass, band stop or another digital filter. These transformations are given below.

Low pass to Low pass	**Low pass to high pass**
$z^{-1} \to \dfrac{z^{-1}-\alpha}{1-\alpha z^{-1}}$ $where\ \alpha = \dfrac{\sin\left[(\omega_p - \omega_p^{'})/2\right]}{\sin\left[(\omega_p + \omega_p^{'})/2\right]}$ $\omega_p = passband\ \ frequency\ of\ lowpass\ filter$ $\omega_p^{'} = passband\ \ frequency\ of\ new\, lowpass\ fi$	$z^{-1} \to -\left[\dfrac{z^{-1}+\alpha}{1+\alpha z^{-1}}\right]$ $where\ \alpha = -\dfrac{\cos\left[(\omega_p^{'} + \omega_p)/2\right]}{\cos\left[(\omega_p^{'} - \omega_p)/2\right]}$ $\omega_p = passband\ \ frequency\ of\ lowpass\ fil$ $\omega_p^{'} = passband\ \ frequency\ of\ highpass\ fi$
Low pass to Band pass	**Low pass to Band Stop**
$z^{-1} \to \dfrac{-\left(z^{-2} - \dfrac{2\alpha k}{1+k} z^{-1} + \dfrac{k-1}{k+1}\right)}{\dfrac{k-1}{k+1} z^{-2} - \dfrac{2\alpha k}{1+k} z^{-1} + 1}$ $where\ \alpha = \dfrac{\cos\left[(\omega_u + \omega_l)/2\right]}{\cos\left[(\omega_u - \omega_l)/2\right]}$ $k = \cot\left[\dfrac{\omega_u - \omega_l}{2}\right]\tan\dfrac{\omega_p}{2}$ $\omega_u = upper\ cutoff\ \ frequency$ $\omega_l = lower\ cutoff\ \ frequency$	$z^{-1} \to \dfrac{z^{-2} - \dfrac{2\alpha}{1+k} z^{-1} + \dfrac{1-k}{1+k}}{\dfrac{1-k}{1+k} z^{-2} - \dfrac{2\alpha}{1+k} z^{-1} + 1}$ $where\ \alpha = \dfrac{\cos\left[(\omega_u + \omega_l)/2\right]}{\cos\left[(\omega_u - \omega_l)/2\right]}$ $k = \tan\left[\dfrac{\omega_u - \omega_l}{2}\right]\tan\dfrac{\omega_p}{2}$ $\omega_u = upper\ cutoff\ \ frequency$ $\omega_l = lower\ cutoff\ \ frequency$

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<u>Analog Domain:</u>

The frequency transformation can be used to design on LPF with different pass band frequency HPF,BPF and BSF from a normalized Low pass filter Ω_C=1 rad/sec.

Low pass to Low pass	**Low pass to high pass**
$S \to \dfrac{S}{\Omega_C}$	$S \to \dfrac{\Omega_C}{S}$
Low pass to band pass	**Low pass to band stop**
$s \to \dfrac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$ $\Omega_r = \min\{\lvert A\rvert, \lvert B\rvert\}$ $A = \dfrac{-\Omega_1^{\ 2} + \Omega_l \Omega_u}{\Omega_1(\Omega_u - \Omega_l)}$ $B = \dfrac{\Omega_2^{\ 2} - \Omega_l \Omega_u}{\Omega_2(\Omega_u - \Omega_l)}$	$s \to \dfrac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$ $\Omega_r = \min\{\lvert A\rvert, \lvert B\rvert\}$ $\qquad A = \dfrac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^{\ 2} + \Omega_l \Omega_u}$ $B = \dfrac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^{\ 2} + \Omega_l \Omega_u}$

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H:W: **1. Design a digital chebyshev filter**
$$\frac{1}{\sqrt{2}} \le H\left(e^{j\omega}\right) \le 1 \quad \text{for } 0 \le \omega \le 0.2\pi$$
$$0 \le \left|H\left(e^{j\omega}\right) \le 0.1 \quad \text{for } 0.5\pi \le \omega \le \pi\right|$$

by using bilinear transformation and assume period T=1 sec.

Ans :

$$N = 2; \quad H(s) = \frac{0.212}{s^2 + 0.4172s + 0.3} \quad ; \; H(z) = \frac{0.0413\left(1 + z^{-1}\right)^2}{1 - 1.44z^{-1} + 0.675z^{-2}}$$

2. Enumerate the various steps involved in the design of low pass digital Butterworth IIR filter.

$$0.8 \le H\left(e^{j\omega}\right) \le 1 \quad \text{for } 0 \le \omega \le 0.2\pi$$
$$\left\|H\left(e^{j\omega}\right)\right| \le 0.2 \quad \text{for } 0.32\pi \le \omega \le \pi\right|$$

Design Butterworth digital filter using impulse invariant transformation.

Ans:

$$N = 4; \quad H(s) = \frac{0.2084}{\left(s^2 + 0.5171s + 0.4565\right)\left(s^2 + 1.2485s + 0.4565\right)} \quad ;$$

$$H(z) = \frac{-0.6242 + 0.2747z^{-1}}{1 - 0.253z^{-1} + 0.5963z^{-2}} + \frac{0.6242 - 0.1168z^{-1}}{1 - 1.03582z^{-1} + 0.2869z^{-2}}$$

3. Design a chebyshev low pass filter with the specifications $\alpha s = 1dB$ ripple in the pass band $0 \le \omega \le 0.2\pi, \alpha s = 15dB$ ripple in the stop band $0.3\pi \le \omega \le \pi$, using (a) Bilinear transformation (b) Impulse invariance.

(a) Bilinear transformation:

Ans:

$$N = 4; \quad H(s) = \frac{0.04381}{\left(s^2 + 0.1814s + 0.4165\right)\left(s^2 + 0.4378s + 0.1180\right)} \quad ;$$

$$H(z) = \frac{0.001836\left(1 + z^{-1}\right)^4}{\left(1 - 1.499z^{-1} + 0.8482z^{-2}\right)\left(1 - 1.5548z^{-1} + 0.6493z^{-2}\right)}$$

(b) Impulse invariance:

Ans:

$$N = 4; \quad H(s) = \frac{0.03834}{\left(s^2 + 0.175s + 0.391\right)\left(s^2 + 0.423s + 0.11\right)} \quad ;$$

$$H(z) = \frac{-0.083 - 0.0245z^{-1}}{1 - 1.49z^{-1} + 0.839z^{-2}} + \frac{0.083 + 0.0238z^{-1}}{1 - 1.56z^{-1} + 0.655z^{-2}}$$

4. Use the backward difference for the derivative to convert the analog low pass filter with system function.

$$H(s) = \frac{1}{s + 2}$$

Ans:
$$H(z) = \frac{1}{3 - z^{-1}}$$

5. For the analog transfer function determine H(z) using impulse invariant technique. Assume T=1sec.

$$H(s) = \frac{1}{(s + 1)(s + 2)}$$
Ans:
$$H(z) = \frac{0.2326z^{-1}}{1 - 0.5032z^{-1} + 0.0498z^{-2}}$$
[May/Jue-2016]

6. Determine H(z) using the impulse invariant technique for the analog transfer function.

$$H(s)=\frac{1}{(s+0.5)(s^2+0.5s+2)}$$

Ans:

$$H(z)=\frac{0.5}{1-0.6065z^{-1}}-0.5\left(\frac{1-0.1385z^{-1}}{1+0.277z^{-1}+0.606z^{-2}}\right)+0.0898\left(\frac{0.7663z^{-1}}{1-0.277z^{-1}+0.606z^{-2}}\right)$$

7. Using bilinear transformation obtain H (z) if $H(s)=\frac{1}{(s+1)^2}$ **and T=0.1s.**

Ans: $H(z)=\frac{0.0476(1+z^{-1})^2}{(1-0.9048z^{-1})^2}$

8.Convert the analog filter with system function $H(s)=\frac{s+0.1}{(s+0.1)^2+9}$ **into a digital IIR filter using bilinear transformation. The digital filter should have a resonant frequency of** $\omega r=\frac{\pi}{4}$. **[Nov/Dec-2015]**

$$H(z)=\frac{1+0.027z^{-1}-0.973z^{-2}}{8.572-11.84z^{-1}+8.177z^{-2}}$$

9. **A digital filter with a 3 dB bandwidth of 0.25π is to be designed from the analog filter whose system response is** $H(s)=\frac{\Omega c}{s+\Omega c}$ **. Use bilinear transformation and obtain H(z). [Nov/Dec-15]**

$$H(z)=\frac{1+z^{-1}}{3.414-1.414z^{-1}}$$

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Prove that $\Omega c=\frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}}=\frac{\Omega_s}{(10^{0.1\alpha_s}-1)^{1/2N}}$

Solution:

$$\left|H(j\Omega)\right|^2=\frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$$we\,know\left|H(j\Omega)\right|^2=\frac{1}{1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}$$

by comparing

$$1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}=1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}$$

$$\left(\frac{\Omega}{\Omega_c}\right)^{2N}=\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}$$

$$= \Omega_c (10^{0.1\alpha_p} - 1)^{1/2N} \left[\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1} \right]^{1/2N}$$

$$\Omega_c = \frac{\Omega_s}{(10^{0.1\alpha_s} - 1)^{1/2N}}$$

$$thus\ \Omega c = \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} = \frac{\Omega_s}{(10^{0.1\alpha_s} - 1)^{1/2N}}$$

$$where \left(\frac{\Omega_p}{\Omega_c} \right)^{2N} = 10^{0.1\alpha_p} - 1$$

$$there\ fore\ \Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} = \frac{\Omega_p}{\varepsilon^{1/N}}$$

$$and \left(\frac{\Omega_s}{\Omega_p} \right)^{2N} = \frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}$$

$$\Omega_s = \Omega_p \left[\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1} \right]^{1/2N}$$