SINGULARITIES – RESIDUES – RESIDUE THEOREM

Zeros of an analytic function

If a function $f(z)$ is analytic in a region R, is zero at a point $z = z_0$ in R, then z_0 is called a zero of $f(z)$.

Simple zero

If $f(z_0) = 0$ and $f'(z_0) \neq 0$, then $z = z_0$ is called a simple zero of $f(z)$ or a zero of the first order.

Zero of order n

If $f(z_0) = f'(z_0) = \cdots = f^{n-1}(z_0) = 0$ and $f^n(z_0) \neq 0$, then z_0 is called zero of order.

Problems based on zeros

Example: 4.27 Find the zeros of $f(z) = \frac{z^2+1}{1-z^2}$

Solution:

The zeros of $f(z)$are given by $f(z) = 0$

$$(i.e.)f(z) = \frac{z^2+1}{1-z^2} = \frac{(z+i)(z-i)}{1-z^2} = 0$$

$$\Rightarrow (z+i)(z-i) = 0$$

$\Rightarrow z = i$ and $-i$ are simple zero.

Example: 4.28 Find the zeros of $f(z) = \sin\frac{1}{z-a}$

Solution:

The zeros are given by $f(z) = 0$

$$(i.e.)\sin\frac{1}{z-a} = 0$$

$$\Rightarrow \frac{1}{z-a} = n\pi, n = \pm 1, \pm 2, \ldots$$

$$\Rightarrow (z-a)n\pi = 1$$

$\therefore$ The zeros are $z = a + \frac{1}{n\pi}, n = \pm 1, \pm 2, ,,$

Example: 4.29 Find the zeros of $f(z) = \frac{\sin z - z}{z^3}$

Solution:

The zeros are given by $f(z) = 0$

$$(i.e.)\frac{\sin z - z}{z^3} = 0$$

$$\Rightarrow \frac{\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} \ldots\right] - z}{z^3} = 0$$

$$\Rightarrow \frac{-\frac{z^3}{3!} + \frac{z^5}{5!} \ldots}{z^3} = 0$$

$$\Rightarrow -\frac{1}{3!} + \frac{z^2}{5!} \ldots = 0$$

But $\lim_{z\to 0}\frac{\sin z - z}{z^3} = -\frac{1}{3!} + 0$

$\therefore f(z)$ has no zeros.

Example: 4.30 Find the zeros of $f(z) = \frac{1-e^{2z}}{z^4}$

Solution:

The zeros are given by $f(z) = 0$

$$(i.e.)\ \frac{1-e^{2z}}{z^4} = 0$$

$$\Rightarrow 1 - e^{2z} = 0$$

$$\Rightarrow e^{2z} = e^{2in\pi}$$

$$(i.e.) 2z = 2in\pi$$

$\Rightarrow z = in\pi, n = 0, \pm 1;\ \pm 2 \ldots$

Singular points

A point $z = z_0$ at which a function $f(z)$ fails to be analytic is called a singular point or singularity of $f(z)$.

Example: Consider $f(z) = \frac{1}{z-5}$

Here, $z = 5$, is a singular point of $f(z)$

Types of singularity

A point $z = z_0$ is said to be isolated singularity of $f(z)$ if

(i) $f(z)$ is not analytic at $z = z_0$

(ii) There exists a neighbourhood of $z = z_0$ containing no other singularity

Example: $f(z) = \frac{z}{z^2-1}$

This function is analytic everywhere except at $z = 1, -1$

$\therefore z = 1, -1$ are two isolated singular points.

When $z = z_0$ is an isolated singular point of $f(z)$, it can expand $f(z)$ as a Laurent's series about $z = z_0$

Thus

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=0}^{\infty} b_n (z - z_0)^{-n}$$

Note: If $z = z_0$ is an isolated singular point of a function $f(z)$, then the singularity is called

(i) a removable singularity (or)

(ii) a pole (or)

(iii) an essential singularity

According as the Laurent's series about $z = z_0$ of $f(z)$ has

(i) no negative powers (or)

(ii) a finite number of negative powers (or)

(iii) an infinite number of negative powers

Removable singularity

If the principal part of $f(z)$ in Laurent's series expansion contains no term $(i.e.) b_n = 0$ for all n, then the singularity $z = z_0$ is known as the removable singularity of $f(z)$

$$\therefore f(z) = \sum_{n=0}^{\infty} a_n (z - z_o)^n$$

(OR)

A singular point $z = z_0$ is called a removable singularity of $f(z)$, if $\lim_{z \to z_0} f(z)$ ezists finitely

Example: $f(z) = \frac{\sin z}{z}$

$$\frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} \dots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!}$$

There is no negative powers of z.

$\therefore z = 0$ is a removable singularity of $f(z)$.

Poles

If we can find the positive integer n such that $\lim_{z \to z_0} (z - z_0)^n f(z) \neq 0$, then $z = z_0$ is called a pole of order n for $f(z)$.

(or)

If $\lim_{z \to z_0} f(z) = \infty, then\ z = z_0$ is a pole of $f(z)$

Simple pole

A pole of order one is called a simple pole.

Example: $f(z) = \frac{1}{(z-1)^2(z+2)}$

Here $z = 1$ is a pole of order 2

$z = 2$ is a pole of order 1.

Essential singularity

If the principal part of $f(z)$ in Laurent's series expansion contains an infinite number of non zero terms, then $z = z_0$ is known as an essential singularity.

Example: $f(z) = e^{1/z} = 1 + \frac{\frac{1}{z}}{1!} + \frac{\left(\frac{1}{z}\right)^2}{\mathbf{2}!} + \cdots$ has $z = 0$ as an essential singularity since, $f(z)$ is an infinite series of negative powers of z.

$f(z) = e^{\frac{1}{z}-4}$ has $z = 4$ an essential singularity

Note: The removable singularity and the poles are isolated singularities. But, the essential singularity is either an isolated or non-isolated singularity.

Entire function (or) Integral function

A function $f(z)$ which is analytic everywhere in the finite plane (except at infinity) is called an entire function or an integral function.

Example: $e^z, \sin z\,, \cos z$ are all entire functions.

Problems Based on Singularities

Example: 4.31 What is the nature of the singularity $z = 0$ of the function $f(z) = \frac{sinz-z}{z^3}$

Solution:

Given $f(z) = \frac{sinz-z}{z^3}$

The function $f(z)$is not defined at $z = 0$

By L' Hospital's rule.

$$\lim_{z\to 0} \frac{sinz-z}{z^3} = \lim_{z\to 0} \frac{cosz-1}{3z^2}$$

$$= \lim_{z\to 0} \frac{-sinz}{6z}$$

$$= \lim_{z\to 0} -\frac{cosz}{6z} = \frac{-1}{6}$$

Since, the limit exists and is finite, the singularity at $z = 0$ is a removable singularity.

Example: 4.32 Classify the singularities for the function$\boldsymbol{f(z) = \frac{z-sinz}{z}}$

Solution:

Given $f(z) = \frac{\boldsymbol{z-sinz}}{\boldsymbol{z}}$

The function $f(z)$is not defined at $z = 0$

But by L' Hospital's rule.

$$\lim_{z\to 0} \frac{z-sinz}{z} = \lim_{z\to 0} 1 - cosz \quad = 1 - 1 = 0$$

Since, the limit exists and is finite, the singularity at $z = 0$ is a removable singularity.

Example: 4.33 Find the singularity of$\boldsymbol{f(z) = \frac{e^{1/z}}{(z-a)^2}}$

Solution:

Given $f(z) = \frac{e^{1/z}}{(z-a)^2}$

Poles of $f(z)$ are obtained by equating the denominator to zero.

$$(i.e.)(z-a)^2 = 0$$

$\Rightarrow z = a$ is a pole of order 2.

Now, Zeros of $f(z)$

$$\lim_{z\to 0} \frac{e^{1/z}}{(z-a)^2} = \frac{\infty}{a^2} = \infty \neq 0$$

$\Rightarrow$ $z = 0$is a removable singularity.

$\therefore$ $f(z)$ has no zeros.

Example: 4.34 Find the kind of singularity of the function $\boldsymbol{f(z) = \frac{cot\pi z}{(z-a)^2}}$

Solution:

Given $f(z) = \frac{cot\pi z}{(z-a)^2}$

$$= \frac{\cos \pi z}{sin\pi z(z-a)^2}$$

Singular points are poles, are given by

$$\Rightarrow sin\pi z(z-a)^2 = 0$$

$$(i.e.)sin\pi z = 0, (z-a)^2 = 0$$

$\pi z = n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$

$(i.e.) z = n$

$z = a$ is a pole of order 2

Since $z = n, n = 0, \pm 1, \pm 2, \ldots$

$z = \infty$ is a limit of these poles.

$\therefore z = \infty$ is non- isolated singularity.

Example: 4.35 Find the singular point of the function $f(z) = sinz\frac{1}{z-a}$. State nature of singularity.

Solution:

Given $f(z) = sinz\frac{1}{z-a}$

$z = a$ is the only singular point in the finite plane.

$$sinz\frac{1}{z-a} = \frac{1}{z-a} - \frac{1}{3!(z-a)^3} + \frac{1}{5!(z-a)^5} - \cdots$$

$z = a$ is an essential singularity

It is an isolated singularity.

Example: 4.36 Identify the type of singularity of the function $f(z) = sin\left(\frac{1}{1-z}\right)$.

Solution:

$z = 1$ is the only singular point in the finite plane.

$z = 1$ is an essential singularity

It is an isolated singularity.

Example: 4.37 Find the singular points of the function $f(z) = \left(\frac{1}{sin\frac{1}{z-a}}\right)$, state their nature.

Solution:

$f(z)$has an infinite number of poles which are given by

$$\frac{1}{z-a} = n\pi, n = \pm 1, \pm 2, \ldots$$

$(i.e.) z - a = \frac{1}{n\pi}\ ; z = a + \frac{1}{n\pi}$

But $z = a$ is also a singular point.

It is an essential singularity.

It is a limit point of the poles.

So, It is an non - isolated singularity.

Example: 4.38 Classify the singularity of $f(z) = \frac{tanz}{z}$.

Solution:

Given $f(z) = \frac{tanz}{z}$

$$= \frac{z + \frac{z^3}{3} + \frac{2z^5}{15} + \ldots}{z}$$

$$= 1 + \frac{z^2}{3} + \frac{2z^4}{15} + \ldots$$

$$\lim_{z\to 0}\frac{tanz}{z} = 1 \neq 0$$

$\Rightarrow$ $z = 0$is a removable singularity of $f(z)$.

Example: 4.39 Find the residue of $\frac{1-e^z}{z^4}$ at $z = 0$

Solution:

$$\text{Given } f(z) = \frac{1-e^z}{z^4} = \frac{1-\left[1+\frac{2z}{1!}+\frac{(2z)^2}{2!}+\frac{(2z)^3}{3!}+\frac{(2z)^4}{4!}+\ldots\right]}{z^4}$$

$$= \frac{-\left[\frac{2}{1!}+\frac{4z}{2!}+\frac{8z^2}{3!}+\frac{16z^3}{4!}+\ldots\right]}{z^4}$$

Here, $z = 0$ is a pole of order 3

$$[Res\, f(z), z = 0] = \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}[(z)^3 f(z)]$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\left[-\left[\frac{2}{1!}+2z+\frac{4z^2}{3}+\frac{2z^3}{3}+\ldots\right]\right]$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left[-\left[2+\frac{8}{3}z+\frac{6z^2}{3}+\ldots\right]\right]$$

$$= \frac{1}{2}\lim_{z\to 0}\left[-\left(\frac{8}{3}+\frac{12}{3}z+\cdots\right)\right]$$

$$= \frac{1}{2}\left(\frac{-8}{3}\right) = \frac{-4}{3}$$

Example: 4.40 Find the residue of $f(z) = tanz$ at $z = \frac{\pi}{2}$

Solution:

$$\left[\text{Res f(z)}, \text{z} = \frac{\pi}{2}\right] = \lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right)\text{ tanz}$$

$$= \lim_{z\to\frac{\pi}{2}}\frac{z-\frac{\pi}{2}}{\cot z} \quad \left[\frac{0}{0}\right] \text{ form}$$

$$= \lim_{z\to\frac{\pi}{2}}\frac{1}{-\text{cosec}^2 z} = -1[\text{By L'Hospital rule}]$$

Residue

The residue of $f(z)$ at $z = z_0$ is the coefficient of $\frac{1}{z-z_0}$ in the Laurent series of $f(z)$ about $z = z_0$

Evaluation of Residues

(i) If $z = z_0$ is a pole of order one (simple pole) for $f(z)$, then

$$[Res\, f(z), z = z_0] = \lim_{z\to z_0}(z-z_0)\, f(z).$$

(ii) If $z = z_0$ is a pole of order n for $f(z)$, then

$$[Res\, f(z), z = z_0] = \lim_{z\to z_0}\frac{1}{(n-1)!}\frac{d^{\,n-1}}{dz^{n-1}}(z-z_0)^n\, f(z)$$

Problems based on Residues

Example: 4.41 Calculate the residue of $f(z) = \frac{e^{2z}}{(z+1)^2}$ at its pole.

Solution:

Given $f(z) = \frac{e^{2z}}{(z+1)^2}$ Here, $z = -1$ is a pole of order 2.

We know that,

$$[Res\, f(z), z = z_0] = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{\,m-1}}{dz^{m-1}} (z - z_0)^m \; f(z)$$

Here, $m = 2$

$$[Res\, f(z), z = -1] = \lim_{z \to -1} \frac{1}{1!} \frac{d}{dz} (z+1)^2 \frac{e^{2z}}{(z+1)^2}$$

$$= \lim_{z \to -1} \frac{d}{dz} [e^{2z}] \quad = \lim_{z \to -1} 2[e^{2z}] = 2\, e^{-2}$$

Example: 4.42 Find the residues at $z = 0$ of the function (i) $f(z) = e^{1/z}$ (ii) $f(z) = \frac{sinz}{z^4}$

(iii) $f(z) = zcos\frac{1}{z}$

Solution:

The residues are the coefficients of $\frac{1}{z}$ in the Laurent's expansions of $f(z)$ about $z = 0$

(i) $e^{1/z} = 1 + \frac{\left(\frac{1}{z}\right)}{1!} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \cdots$

$$= 1 + \frac{1}{1!}\left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \cdots$$

$[Res\, f(z), 0\,]$ = coefficient of $\frac{1}{z}$ in Laurent's expansion.

$[Res\, f(z), 0\,] = \frac{1}{1!} = 1$by definition of residue.

(ii) $f(z) = \frac{sinz}{z^4} = \frac{1}{z^4}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right] = \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z} + \frac{z^5}{5!} - \;\ldots$

$[Res\, f(z), 0\,]$ = coefficient of $\frac{1}{z}$ in Laurent's expansion.

$[Res\, f(z), 0\,] = -\frac{1}{3!} = -\frac{1}{6}$ by definition of residue.

(iii) $f(z) = zcos\frac{1}{z} = z\left[1 - \frac{1}{2!}\frac{1}{z^2} + \frac{1}{4!}\frac{1}{z^4} - \cdots\right]$

$$= z - \frac{1}{2!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^3} - \cdots$$

$[Res\, f(z), 0\,]$ = coefficient of $\frac{1}{z}$ in Laurent's expansion.

$[Res\, f(z), 0\,] = -\frac{1}{2!} = -\frac{1}{2}$

Example: 4.43 Find the residue of $z^2 sin\left(\frac{1}{z}\right)$ at $z = 0$

Solution:

Let $f(z) = z^2 sin\left(\frac{1}{z}\right) = z^2\left[\frac{\left(\frac{1}{z}\right)}{1!} - \frac{\left(\frac{1}{z}\right)^3}{3!} + \cdots\right] = \frac{z}{1!} - \frac{1}{6z} + \cdots$

$[Res\, f(z), 0\,]$ = coefficient of $\frac{1}{z}$ in Laurent's expansion.

$$= -\frac{1}{6}$$

Example: 4.44 Find the residue of the function $f(z) = \frac{4}{z^3(z-2)}$ at a simple pole.

Solution:

Here, $z = 2$ is a simple pole.

$$[Res\, f(z), z = 2] = \lim_{z \to 2} (z-2) \frac{4}{z^3(z-2)}$$

$$= \lim_{z \to 2} \frac{4}{z^3} = \frac{4}{8} = \frac{1}{2}$$

Example: 4.45 Find the residue of $\frac{1-e^{-z}}{z^3}$ at $z = 0$

Solution:

Given $f(z) = \frac{1-e^{-z}}{z^3} = \frac{1-\left[1-\frac{z}{1!}+\frac{(z)^2}{2!}-\frac{(z)^3}{3!}+\frac{(z)^4}{4!}-\ \dots\right]}{z^3}$

$$= \frac{\left[1-\frac{z}{2!}+\frac{z^2}{3!}-\frac{z^3}{4!}+\dots\right]}{z^2}$$

Here, $z = 0$ is a pole of order 2.

$$[Res\, f(z), z = 0] = \frac{1}{1!} \lim_{z \to 0} \frac{d}{dz} [(z)^2 f(z)]$$

$$= \lim_{z \to 0} \frac{d}{dz} \left[\left[1 - \frac{z}{2!} + \frac{z^2}{3!} - \frac{z^3}{4!} + \dots\right]\right]$$

$$= \lim_{z \to 0} \left[\frac{-1}{2!} + \frac{2z}{3!} - \frac{3z^2}{4!} + \dots\right]$$

$$= \frac{-1}{2!} = -\frac{1}{2}$$