COULOMB'S LAW

Coulomb's law is formulated in 1785 by French colonel , Charles Augustin de Coulomb. It deals with a force exerts on a point charge due to another point charge . Charges are generally measured in coulombs , one coulomb = 6 X 10^{18} electrons; one electron e = -1.609 X 10^{-19}.

STATEMENT :

The force between two point charges Q1 and Q 2 is

- Along the line joining them
- Directly proportional to the product Q1Q2 of the charges
- Inversely proportional to the square of the distance R between them.

Mathematically expressed as $F = \frac{K\,Q1\,Q2\,a_R}{R^2}$ (1)

K → Proportionality constant , Q1,Q2 → charges in coulombs (C) ,

R → distance in meters (m) , F → Force in newton (N)

$$K = \frac{1}{4\pi\epsilon_0} = 9\,X\,10^9\ m/F$$

ϵ_0 → Permittivity of free space (in farads per meter)

$$\epsilon_0 = 8.854\,X10^{-12} \approx \frac{10^{-9}}{36\pi}\frac{F}{m}$$

Substitute K in equation (1) , Equation (1) becomes

$$F = \frac{Q1\,Q2\,a_R}{4\pi\epsilon_0 R^2} \qquad (2)$$

If the point charge Q1 and Q2 have a position vectors r_1 and r_2 , then the force F is represented as F_{12}

$$F_{12} = \frac{Q1\,Q2}{4\pi\epsilon_0 R^2}\,a_{R_{12}} \qquad (3)$$

Where $R_{12} = r_2 - r_1$ (4)

Magnitude $R = |\boldsymbol{R_{12}}|$ (5)

Unit vector $a_{R_{12}} = \frac{R_{12}}{R}$ (6)

Substitute eqn (6) in eqn (3)

$$F_{12} = \frac{Q1\, Q2}{4\pi\epsilon_0 R^3} R_{12} \qquad (7)$$

Substitute eqn (4) in eqn (7)

$$F_{12} = \frac{Q1\, Q2(r_2 - r_1)}{4\pi\epsilon_0 |r_2 - r_1|^3} \qquad (8)$$

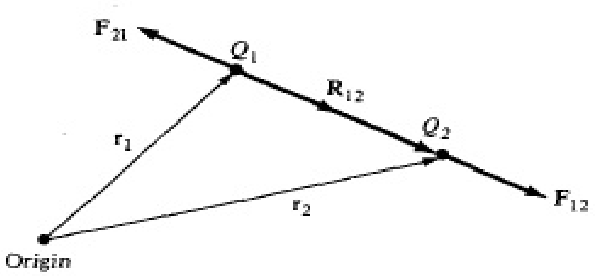


Fig 18 : Coulomb vector force on point charges Q1 and Q2

1. The force on Q1 due to Q2 is $F_{21} = |F_{12}| a_{R_{21}} = |F_{12}|(-a_{R_{12}})$

$F_{21} = - F_{12}$ (8)

$a_{R_{21}} = - a_{R_{12}}$

Fig 19: a,b, - Like charges repel ; c- Unlike charges attract.

2. Like charges repel each other while unlike charges attract.
3. The distance R between the charges Q1 and Q2 must be large compared with the dimensions of Q1 and Q2.
4. Q1 and Q2 must be static (at rest).
5. The signs of Q1 and Q2 must be taken into account in eqn (3).

Force due to N – point charges : (Principle of Super position)

If more than two point (N point) charges Q_1, Q_2, Q_3 , …. Q_N are located with position vectors r_1, r_2, r_3, ……..r_N , the force on charge Q located at point r is the vector sum of the forces exerted on q by each of the charges Q_1, Q_2, Q_3 , …. Q_N .

$$F = \frac{QQ_1(r-r_1)}{4\pi\epsilon_0|r-r_1|^3} + \frac{QQ_2(r-r_2)}{4\pi\epsilon_0|r-r_2|^3} + \ldots \frac{QQ_N(r-r_N)}{4\pi\epsilon_0|r-r_N|^3} \qquad (9)$$

The above equation can also be written as

$$F = \frac{Q}{4\pi\epsilon_0}\sum_{k=1}^{N}\frac{Q_k(r-r_k)}{|r-r_k|^3} \qquad (10)$$

Electric Field intensity :

The electric field intensity (or electric field strength) E is the force per unit charge when placed in the electric field.

$$E = \lim_{Q\to 0}\frac{F}{Q} \qquad (11)$$

$$\mathrm{E} = \frac{\mathrm{F}}{\mathrm{Q}} \qquad (12)$$

The electric field intensity E is in the direction of the force F and is measured in newtons/ coulomb or volts / meter. The electric field intensity at point r due to point charge located at r' is obtained from eqn (8) and eqn (12)

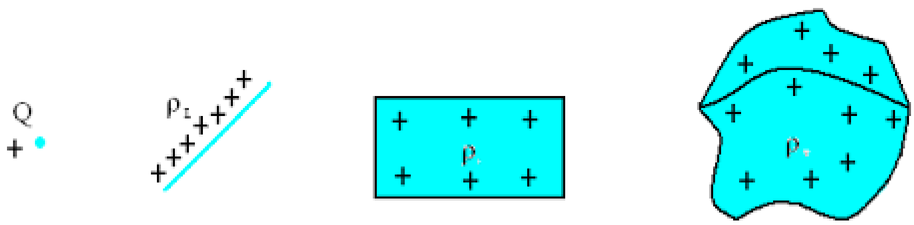
$$E = \frac{Q}{4\pi\epsilon_0 R^2}\, a_R = \frac{Q\,(r-r')}{4\pi\epsilon_0|r-r'|^3} \qquad (13)$$

For N point charges Q_1, Q_2, Q_3 , …. Q_N are located with position vectors r_1, r_2, r_3, ……..r_N the electric field intensity at point r is obtained from eqn (10) and (12)

$$E = \frac{Q_1(r-r_1)}{4\pi\epsilon_0|r-r_1|^3} + \frac{Q_2(r-r_2)}{4\pi\epsilon_0|r-r_2|^3} + \ldots \frac{Q_N(r-r_N)}{4\pi\epsilon_0|r-r_N|^3} \qquad (14)$$

$$E = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{N}\frac{Q_k(r-r_k)}{|r-r_k|^3} \qquad (15)$$

Electric fields due to continuous charge distributions



Point charge | Line charge | Surface charge | volume charge

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Fig 20. Various charge distributions and charge elements

- Charge distribution is continuous along a line, surface & volume.

$dQ \rightarrow$ charge element

Total charge $dQ = \rho_L \, dl$

$$Q = \int \rho_L \, dl \ \ (line\ charge)$$

$$dQ = \rho_s \, ds \ (Surface\ charge)$$

$$Q = \int_s \rho_s \, ds$$

Volume charge

$$dQ = \rho_v \, dv$$

$$Q = \int_v \rho_v \, dv$$

The electric field intensity due to $\rho_{L,}\rho_S$, ρ_v

$$E = \frac{F}{Q} = \frac{Q\,(r - r')}{4\pi\varepsilon_0 |r - r'|^3} = \frac{Q\, a_R}{4\pi\varepsilon_0 R^2}$$

$$E = \frac{\int_L P_L \, dL}{4\pi\varepsilon_0 R^2} a_R \text{ (Line charge)}$$

$$E = \int_s \frac{P_S \, ds}{4\pi\varepsilon_0 R^2} a_R \text{ (Surface charge)}$$

$$E = \int_v \frac{P_v \, dv}{4\pi\varepsilon_0 R^2} a_R \text{ (Volume Charge)}$$

Electric field intensity of Line charge (Finite and infinite Line Charge)

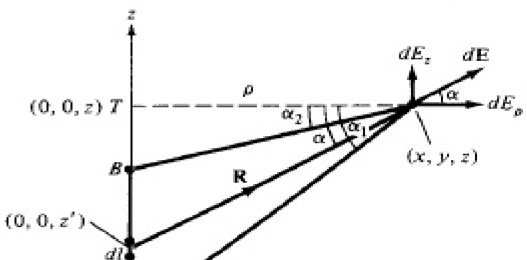

Fig 21: Evaluation of the E field due to a line charge

Line charge with uniform distribution from A to B along z axis

The charge element dQ associated with element $dl = dz$

$$dQ = \rho_L\, dL = \rho_L\, dz \qquad \ldots\ldots\ldots..\ (1)$$

Total charge $Q = \int_L \rho_L\, dL = \int_{zA}^{zB} \rho_L\, dz$............ (2)

The electric field intensity E at point $P(x, y, z)$

$$dl = dz'$$

$$R = (x, y, z) - (0,0, z') \qquad \rho = R\cos\alpha$$

$$= xax + yay + (z - z')az \qquad z - z' = R\sin\alpha$$

(or) $R = \rho a\rho + (z - z')az$

$$R^2 = |R^2| = x^2 + y^2 + (z - z')^2 = \rho^2 + (z - z')^2$$

$$\frac{a_R}{R^2} = \frac{R}{|R^3|} = \frac{\rho a\rho + (z - z')az}{|\rho^2 + (z - z')^2|^{3/2}} \qquad \ldots\ldots\ldots\ (3)$$

$$E = \frac{\int_L \rho_L\, dL}{4\pi\varepsilon_0 R^2} a_R$$

$$E = \frac{\rho_L}{4\pi\varepsilon_0} \int \frac{\rho a\rho + (z - z')az}{(\rho^2 + (z - z')^2)^{3/2}} dz' \qquad \ldots\ldots\ldots\ldots\ (4)$$

From fig 21

$$R = [\rho^2 + (z - z')^2]^{1/2} = \rho\sec\alpha$$

$$z' = OT - \rho\tan\alpha$$

$$\tan\alpha = \frac{z - z'}{\rho}$$

$$-dz' = \rho\sec^2\alpha\, d\alpha$$

$$dz' = -\rho \sec^2 \alpha \, d\alpha$$

(4) $\Longrightarrow$ becomes

$$E = \frac{-\rho_L}{4\pi\varepsilon_0} \int_{\alpha_1}^{\alpha_2} \frac{\rho \sec^2 \alpha \, [R \cos \alpha_\rho + R \sin \alpha \, az]}{(\rho^2 sec^2 \, \alpha) \, R} d\alpha$$

$$E = \frac{-\rho_L}{4\pi\varepsilon_0 \rho} \int_{\alpha_1}^{\alpha_2} [\, Cos \, \alpha \, a\rho + \sin \alpha \, az] \;\; d\alpha$$

$$E = \frac{-\rho_L}{4\pi\varepsilon_0 \rho} [\, +sin \;\; \alpha \; a\rho - \cos \alpha \;\; \alpha z]_{\alpha_1}^{\alpha_2}$$

$$E = \frac{-\rho_L}{4\pi\varepsilon_0 \rho} \;\; [\, +[sin \;\; \alpha_2 - \sin \alpha_1] a\rho - \;\; [Cos\alpha_2 - \;\; cos \; \alpha_1] \; az \,]$$

Finite line charge

$$E = \frac{\rho_L}{4\pi\varepsilon_0 \rho} [-[sin \;\; \alpha_2 - \sin \alpha_1]] a\rho + \;\; [Cos\alpha_2 - \;\; cos \; \alpha_1] \; az \,]$$

…. (A)

Infinite line charge

Point B is at $(0,0,\propto)$ and A at $(0,0,-\propto) \alpha_1 = \frac{\pi}{2} \alpha_2 = \frac{-\pi}{2}$

Eqn (A) $\Rightarrow$

$$E \;\; = \frac{\rho_L}{4\pi\varepsilon_0 \rho} \left[-(-sin \;\; \frac{\pi}{2} - \sin \frac{\pi}{2} \right] a\rho + \;\; 0]$$

$$E \;\; = \frac{\rho_L}{4\pi\varepsilon_0 \rho} \;\; 2a\rho = \;\; \frac{\rho_L}{2\pi\varepsilon_0 \rho} \;\; a\rho \text{ v/m}$$

Electric field intensity of an infinite sheet of charge:

Infinite sheet of charge in xy plane with charge density ρs. Total charge with an elemental area (ds) is

$$dQ = \rho_s . \, ds \longrightarrow \quad (1)$$

Integrate eqn (1)

$$Q = \int_s \rho_s ds \longrightarrow \qquad (2)$$

The electric field intensity is expressed in general as

$$E = \frac{Q}{4\pi\varepsilon_0 R^2} a_R \longrightarrow$$

(A)

For a surface charge sub (2) in (A)

$$E = \int_s \frac{\rho_s \, ds}{4\pi\varepsilon_0 \, R^2} a_R \longrightarrow \qquad (3)$$

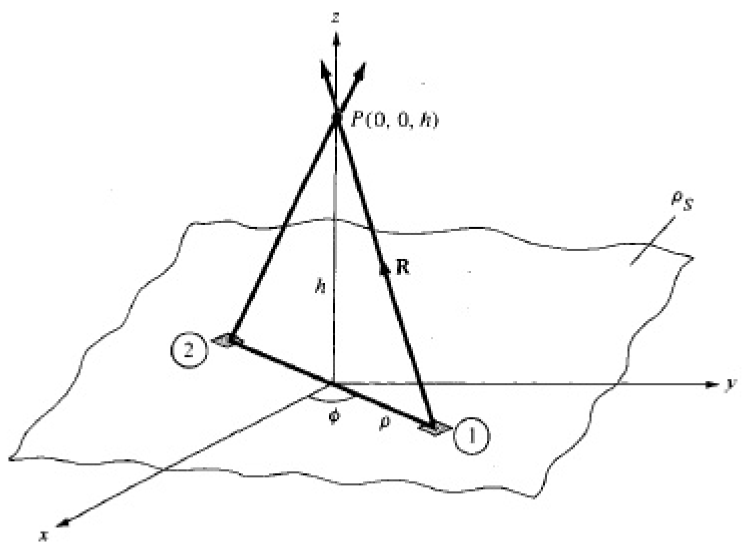

Fig22: Evaluation of electric field 'E' due to an infinite sheet of charge.

Find ds. aR and R then substitute in equation (3)

$$R = \rho\,(-a_\rho) + h\,a_z$$

$$R = -\rho\,a_\rho + h\,a_z \longrightarrow \qquad (4)$$

$$|R| = \sqrt{\rho^2 + h^2} \longrightarrow$$

(4a)

The surface is in xy plane and is given as $ds = \rho . d\Phi . d\rho \longrightarrow$ (5)

$$a_R = \frac{R}{|R|} \longrightarrow$$

(6)

sub eqn (6), (5) ,(4) and (4a) in eqn (3)

$$E = \int_0^{2\pi}\int_0^{\infty} \frac{(-\rho a\rho + h\ az)\rho_S}{4\varepsilon_0\pi(\rho^2+h^2)(\sqrt{\rho^2+h^2})} .\rho d\phi . d\rho \longrightarrow$$
(7)

$0 < \Phi < 2\pi$, $0 < \rho < \infty$

E has only 'z' component $-\rho a\rho + haz = h\ az \longrightarrow$ (7a)

Sub eqn (7a) in (7)

$$=\int_0^{2\pi}\int_0^{\infty} \frac{\rho_S haz.\rho d\phi.d\rho}{4\pi\ \varepsilon_0(\rho^2+h^2\ )\sqrt{\rho^2+h^2}}$$

Integrate with respect to Φ

$$= \frac{\rho_S}{4\pi\varepsilon_0}\int_0^{\infty} \frac{\rho haz.d\rho}{(\rho^2+h^2\ )\sqrt{\rho^2+h^2}} [\Phi]_0^{2\pi}$$

$$= \frac{\rho_S}{4\pi\varepsilon_0}\int_0^{\infty} \frac{\rho haz.d\rho}{(\rho^2+h^2\ )\sqrt{\rho^2+h^2}} [\ 2\pi - 0]$$

$$= \frac{2\pi\rho_S h}{4\pi\varepsilon_0}\int_0^{\infty} \frac{\rho az.d\rho}{(\rho^2+h^2\ )\sqrt{\rho^2+h^2}} \longrightarrow$$
(8)

Assume $\rho^2 + h^2 = u^2 \longrightarrow$
(9)

Diff. eqn (9)

$$2\rho\ .\ d\rho = 2u\ .\ du \longrightarrow$$
(9a)

$$\rho\ .\ d\rho = u\ .\ du \longrightarrow$$ (9b)

If $\rho = 0$, $u = h$ and $\rho = \infty$, $u = \infty \longrightarrow$ (9c)

substitute eqn (9b) and (9c) in eqn (8)

$$= \frac{\rho_S h}{2\varepsilon_0}\int_h^{\infty} \frac{u.du\ .az}{(u^2)\times\sqrt{u^2}} \longrightarrow$$ (10)

$$= \frac{\rho_S h}{2\varepsilon_0}\int_h^{\infty} \frac{u.du\ .az}{(u^2)\times u}$$

$$= \frac{\rho_S h}{2\epsilon_0}\int_h^{\infty} u^{-2}\ du. az$$

$$= \frac{\rho_s h}{2\varepsilon_0}\left[\frac{u^{-1}}{-1}\right]_h^\infty . \text{ az}$$

$$= \frac{\rho_s h}{2\varepsilon_0}\left[\frac{-1}{u}\right]_h^\infty . \text{ az}$$

$$= \frac{\rho_s h}{2\varepsilon_0}. \text{ az}\left[\frac{-1}{\infty} + \frac{1}{h}\right]$$

$$E = \frac{\rho_s h}{2\varepsilon_0} \times \frac{1}{h} \text{az} = \frac{\rho_s az}{2\varepsilon_0} \longrightarrow$$

(11)

$$\mathbf{E} = \frac{\rho_s}{2\varepsilon_0}\boldsymbol{a_z}$$

E has only 'z' component if charge is in the xy plane equation (11) is valid for h>0, for h<0 az is replaced by –az

Infinite sheet of charge

The electric field due to infinite sheet of charge is everywhere normal to the surface and its magnitude is independent of the distance of a point from plane containing the sheet of charge

$$E = \frac{\rho_s}{2\varepsilon_0} a_n$$

a_n → unit vector normal to the sheet

Electric field due to charged circular ring

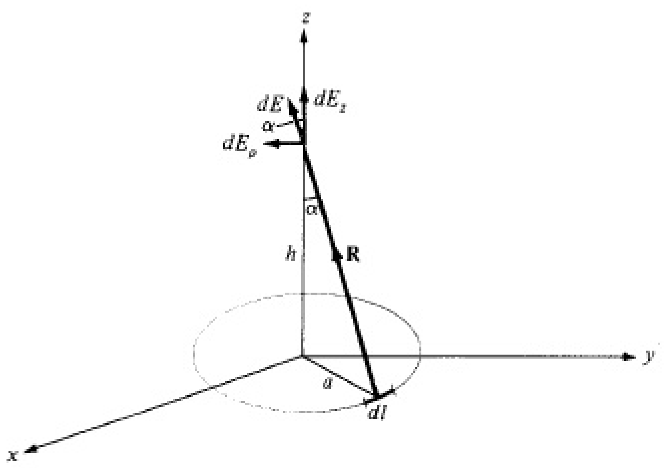

Fig :23 The charged circular ring

The charged circular ring of radius 'ρ' is shown in fig. The ring is placed in xy plane and carrying the charge density is ρ_l c/m [uniformly distributed].

The electric field intensity 'E' is measured at point p which is of distance 'z' from origin.

Consider a small differential length dl on this ring. The total charge on it is

$$dQ = \rho_l \, dl \longrightarrow \quad (1)$$

Integrate eqn (1)

$$Q = \int \rho_l \, dl \longrightarrow \quad (1a)$$

The electric field due to line charge is $E = \int \frac{\rho_l \, dl}{4\pi\varepsilon_0 R^2} . \overline{aR} \longrightarrow \quad (2)$

R = distance between p and dl , dl in Φ direction

$$dl = \rho d\Phi \longrightarrow \quad (3)$$

$$R^2 = \rho^2 + h^2$$

1) 'ρ' is in the direction of –aρ radially inwards (-ρar)
2) h is in the direction a$\bar{z}$ ie h az

$$\bar{R} = -\rho \, a\rho + h \, az \longrightarrow \quad (4)$$

$$|R| = \sqrt{(-\rho)^2 + h^2} = \sqrt{\rho^2 + h^2} \longrightarrow \quad (5)$$

$$aR = \frac{-\rho \, a\rho + h \, az}{\sqrt{\rho^2 + h^2}} \longrightarrow \quad (6)$$

sub eqn (3), (4),(5) and (6) in eqn (2) $0 < \Phi < 2\pi$

$$E = \int_0^{2\pi} \frac{\rho_l \rho d\phi}{4\pi\varepsilon_0(\rho^2+h^2)} \times \frac{-\rho a\rho + haz}{\sqrt{\rho^2+h^2}} \longrightarrow$$

(7)

The radial components of 'E' at point 'p' will be symmetrically placed in the xy plane and are cancel each other

Hence the vector $R = -\rho a\rho + haz$

$$R = h\, a_z \longrightarrow \quad (8) \quad [\text{ since } -\rho a_\rho = 0]$$

Eqn (7) is reduced by

$$= \frac{\rho_l}{4\pi\varepsilon_0} \int_0^{2\pi} \frac{\rho h a_z}{(\rho^2+h^2)^{3/2}} \, . d\Phi$$

$$= \frac{\rho_l}{4\pi\varepsilon_0} \frac{\rho h a_z}{(\rho^2+h^2)^{3/2}} \, [\Phi]_0^{2\pi}$$

$$= \frac{\rho_l}{4\pi\varepsilon_0} \frac{\rho h a_z}{(\rho^2+h^2)^{3/2}} \, . 2\pi$$

$$E = \frac{\rho_l}{2\varepsilon_0} \frac{\rho\, h}{(\rho^2+h^2)^{3/2}} \, . a_z \longrightarrow \quad (9)$$