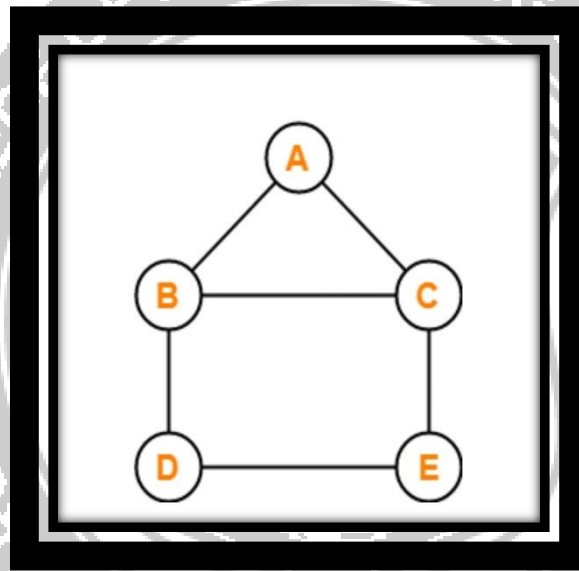


Paths, Reachability and Connectedness:

A path in a graph is a sequence $v_1, v_2, v_3, \dots, v_k$ of vertices each adjacent to the next. In other words, starting with the vertex v_1 , one can travel along edges $(v_1, v_2), (v_2, v_3), \dots$ and reach the vertex v_k .

**Length of the path:**

The number of edges appearing in the sequence of a path is called the length of path.

Cycle or Circuit:

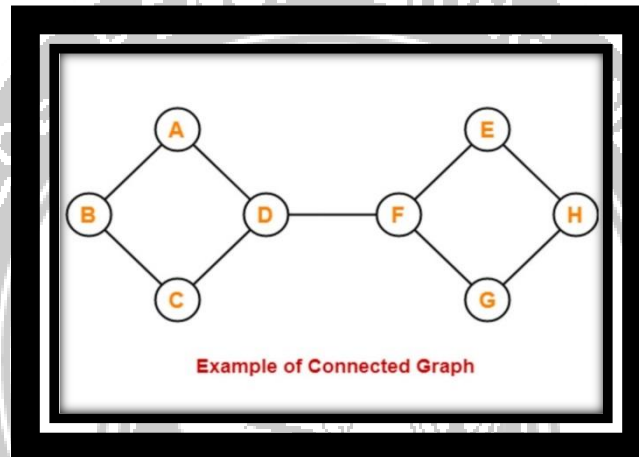
A path which originates and ends in the same node is called a cycle or circuit.

A path is said to be simple if all the edges in the path are distinct.

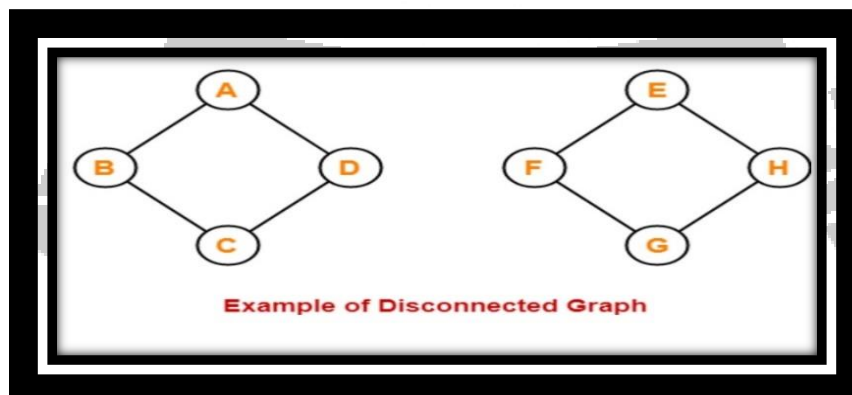
A path in which all the vertices are traversed only once is called an elementary path.

Connected Graph:

An directed graph is said to be connected if any pair of nodes are reachable from one another. That is, there is a path between any pair of nodes.

**Disconnected graph:**

A graph which is not connected is called disconnected graph.



Theorem: 1

If a graph has n vertices and a vertex v is connected to a vertex w , then there exists a path from v to w of length not more than $(n - 1)$.

Proof:

Let $v, u_1, u_2, \dots, u_{m-1}, w$ be a path in G from v to w .

By definition of path, the vertices $v, u_1, u_2, \dots, u_{m-1}$ and w all are distinct.

As G , contains only " n " vertices, it follows that $m + 1 \leq n$

$$\Rightarrow m \leq n - 1$$

Hence the proof.

Theorem: 2

Prove that a simple graph with n vertices must be connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

Proof:

Let G be a simple graph with n vertices and more than $\frac{(n-1)(n-2)}{2}$ edges.

Suppose if G is not connected, then G must have atleast two components. Let it be G_1 and G_2 .

Let V_1 be the vertex set of G_1 with $|V_1| = m$. If V_2 is the vertex set of G_2 , then

$$|V_2| = n - m.$$

Then (i) $1 \leq m \leq n - 1$

(ii) There is no edge joining a vertex of V_1 and a vertex of V_2 .

(iii) $|V_2| = n - m \geq 1$

Now, $|E(G)| = |E(G_1 \cup G_2)|$

$$\begin{aligned} &= |E(G_1)| + |E(G_2)| \\ &\leq \frac{m(m-1)}{2} + \frac{(n-m)(n-m-1)}{2} \\ &= \frac{1}{2} [m^2 - m + n(n-m-1) - m(n-m-1)] \\ &= \frac{1}{2} [n(n-1) - nm - m(n-m-1) + m^2 - m] \\ &= \frac{1}{2} [(n-1)(n-2) + 2(n-1) - 2nm + m^2 + m + m^2 - m] \end{aligned}$$

Adding and Subtracting $2n - 2$

$$\begin{aligned} &= \frac{1}{2} [(n-1)(n-2) + 2n - 2 - 2nm + 2m^2] \\ &= \frac{1}{2} [(n-1)(n-2) + 2n(1-m) + 2(m^2 - 1)] \\ &= \frac{1}{2} [(n-1)(n-2) - 2n(m-1) + 2(m-1)(m+1)] \end{aligned}$$

$$= \frac{1}{2}[(n-1)(n-2) - 2(m-1)(n-m-1)]$$

$$|E(G)| \leq \frac{(n-1)(n-2)}{2}, \text{ Since } (m-1)(n-m-1) \geq 0 \text{ for } 1 \leq m \leq n-1$$

Which is a contradiction as G has more than $\frac{(n-1)(n-2)}{2}$ edges.

Hence G is a connected graph.

Hence the proof.

Theorem: 3

Let G be a simple graph with n vertices. Show that if $\delta(G) \geq \left\lceil \frac{n}{2} \right\rceil$, then G is connected where $\delta(G)$ is minimum degree of the graph G .

Proof:

Let u and v be any two distinct vertices in the graph G .

We claim that there is a $u - v$ path in G .

Suppose uv is not an edge of G . Then, X be the set of all vertices which are adjacent to u and Y be the set of all vertices which are adjacent to v .

Then $u, v \notin X \cup Y$. (Since G is a simple graph)

And hence $|X \cup Y| \leq n - 2$

We have $|X| = \deg(u) \geq \delta(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$ and $|Y| = \deg(v) \geq \delta(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$

Now, $|X| + |Y| \geq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor = n \geq n - 1$

We know that $|X \cup Y| = |X| + |Y| - |X \cap Y|$

$$n - 2 \geq |X \cup Y| \geq n - 1 - |X \cap Y|$$

We have, $|X \cap Y| \geq 1 \Rightarrow X \cap Y \neq \emptyset$

Now, take a vertex $w \in X \cap Y$. Then uvw is a $u - v$ path in G .

Thus for every pair of distinct vertices of G there is a path between them.

Hence G is connected.

Hence the proof.

