3.1 FREQUENCY RESPONSE

The response of a system for the sinusoidal input is called sinusoidal response. The ratio of sinusoidal response to sinusoidal input is called sinusoidal transfer function of the system and in general, it is denoted by, T(jω). The sinusoidal transfer function is the frequency domain representation of the system and so it is also called frequency domain transfer function.

The frequency domain transfer function T(jω) is a complex function of ω. Hence, it can be separated into magnitude function and phase function. Now, the magnitude and phase functions will be real functions of ω and they are called frequency response.

The frequency response can be evaluated for open loop system and closed loop system. The frequency domain transfer function of open loop and closed loop systems can be obtained from the s-domain transfer function by replacing 's' by jω as shown:

Open loop transfer function: $\boldsymbol{G(j\omega) = |G(j\omega)|\angle G(j\omega)}$

Loop transfer function: $\boldsymbol{G(j\omega)H(j\omega) = |G(j\omega)H(j\omega)|\angle G(j\omega)H(j\omega)}$

Closed loop transfer function: $\boldsymbol{M(j\omega) = |M(j\omega)|\angle M(j\omega)}$

The advantages of frequency response analysis are the following:

1. The absolute and relative stability of the closed loop system can be estimated from the knowledge of their open loop frequency response.
2. The practical testing of systems can be easily carried with available sinusoidal signal generators and precise measurement equipments.
3. The transfer function of complicated systems can be determined experimentally by frequency response tests.
4. The design and parameter adjustment of the open loop transfer function of a system for specified closed loop performance is carried out more easily in frequency domain.
5. When the system is designed by the use of frequency response analysis, the effects of noise disturbance and parameter variations are relatively easy to visualize and incorporate corrective measures.
6. The frequency response analysis and designs can be extended to certain non-linear control systems.

The frequency response of a system is a frequency dependent function which expresses how a sinusoidal signal of a given frequency on the system input is transferred through the system. Time-varying signals at least periodical signals – which excite systems, as the reference (set point) signal or a disturbance in a control system or measurement signals which are inputs signals to signal filters, can be regarded as consisting of a sum of frequency components. Each frequency component is a sinusoidal signal having certain amplitude and a certain frequency. (The Fourier series expansion or the Fourier transform can be used to express these frequency components quantitatively.) The frequency response expresses how each of these frequency components is transferred through the system. Some components may be amplified, others may be attenuated, and there will be some phase lag through the system. The frequency response is an important tool for analysis and design of signal filters (as low pass filters and high pass filters), and for analysis, and to some extent, design, of control systems. Both signal filtering and control systems applications are described (briefly) later in this chapter. The definition of the frequency response – which will be given in the next section – applies only to linear models, but this linear model may very well be the local linear model about some operating point of a non-linear model. The frequency response can be found experimentally or from a transfer function model. It can be presented graphically or as a mathematical function.

FREQUENCY DOMAIN SPECIFICATIONS

The performance and characteristics of a system in frequency domain are measured in terms of frequency domain specifications. The requirements of a system to be designed are usually specified in terms of these specifications.

The frequency domain specifications are,

a) Resonant peak, M_r
b) Resonant frequency, ω_r
c) Bandwidth, ω_b
d) Cut-off rate
e) Gain margin, K_g
f) Phase margin, γ

FREQUENCY DOMAIN SPECIFICATIONS OF SECOND ORDER SYSTEM

Resonant peak, M_r

The maximum value of the magnitude of closed loop transfer function is called the resonant peak, M_r. A large resonant peak corresponds to a large overshoot in transient response. Consider the closed loop transfer function of second order system,

$$\frac{C(s)}{R(s)} = M(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The sinusoidal transfer function M(jω) is obtained by letting s=jω.

$$M(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

$$= \frac{\omega_n^2}{-\omega^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

$$= \frac{\omega_n^2}{\omega_n^2\left(-\frac{\omega^2}{\omega_n^2} + 2j\zeta\frac{\omega}{\omega_n} + 1\right)}$$

$$= \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2j\zeta\frac{\omega}{\omega_n}}$$

Let normalized frequency, $u = \left(\frac{\omega}{\omega_n}\right)$,

$$M(j\omega) = \frac{1}{1 - u^2 + 2j\zeta u}$$

Let, M – Magnitude of closed loop transfer function

α – Phase of closed loop transfer function

$$M = |M(j\omega)| = [(1-u^2)^2 + 4\zeta^2u^2]^{-\frac{1}{2}}$$

$$\alpha = \angle M(j\omega) = -\tan^{-1}\frac{2\zeta u}{1-u^2}$$

The resonant peak is the maximum value of M. The condition for maximum value of M can be obtained by differentiating the equation of M with respect to u and letting (dM/du=0) when ($u=u_r$) with normalized frequency, $u_r = \frac{\omega_r}{\omega}$.

On differentiating 'M' with respect to 'u', we get,

$$\frac{dM}{du} = \frac{d}{du}[1 - u^2 + 2j\zeta u]^{-\frac{1}{2}}$$

$$= -\frac{1}{2}[1-u^2+2j\zeta u]^{-\frac{3}{2}}[2(1-u^2)(-2u)+8\zeta^2 u]$$

$$= -\frac{[-4u(1-u^2)+8\zeta^2 u]}{2[(1-u^2)^2+4\zeta^2u^2]^{\frac{3}{2}}}$$

$$= -\frac{[4u(1-u^2)-8\zeta^2 u]}{2[(1-u^2)^2+4\zeta^2u^2]^{\frac{3}{2}}}$$

Replacing u by u_r and equating dM/du to zero,

$$\frac{[4u_r(1-{u_r}^2)-8\zeta^2 u_r]}{2[(1-{u_r}^2)^2+4\zeta^2{u_r}^2]^{\frac{3}{2}}} = 0$$

$$4u_r(1-{u_r}^2)-8\zeta^2 u_r = 0$$

$$4u_r-4{u_r}^3-8\zeta^2 u_r = 0$$

$$4u_r-4{u_r}^3 = 8\zeta^2 u_r$$

$$4{u_r}^3 = 4u_r-8\zeta^2 u_r$$

$${u_r}^2 = 1-2\zeta^2$$

$$u_r = \sqrt{1-2\zeta^2}$$

Therefore, the resonant peak occurs when $u_r = \sqrt{1-2\zeta^2}$

On substituting for M with $M=M_r$ and $u=u_r$,

$$M_r = \frac{1}{[(1-{u_r}^2)^2+4\zeta^2{u_r}^2]^{\frac{1}{2}}} = \frac{1}{\left[\left(1-(1-2\zeta^2)\right)^2+4\zeta^2(1-2\zeta^2)\right]^{\frac{1}{2}}}$$

$$= \frac{1}{[4\zeta^4+4\zeta^2-8\zeta^4]^{\frac{1}{2}}} = \frac{1}{[4\zeta^2-4\zeta^4]^{\frac{1}{2}}} = \frac{1}{[4\zeta^2(1-\zeta^2)]^{\frac{1}{2}}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Resonant frequency, ω_r

The frequency at which the resonant peak occurs is called resonant frequency, ω_r. This is related to the frequency of oscillation in the step response and thus it is indicative of the speed of transient response.

Normalized resonant frequency,

$$u_r = \frac{\omega_r}{\omega_n} = \sqrt{1-2\zeta^2}$$

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}$$

Bandwidth, ω_b

The bandwidth is the range of frequencies for which the system normalized gain is more than -3db. The frequency at which the gain is -3db is called cut-off frequency. Bandwidth is usually defined for closed loop system and it transmits the signals whose frequencies are less than the cut-off frequency. The bandwidth is a measure of the ability of a feedback system to reproduce the input signal, noise rejection characteristics and rise time. A large bandwidth corresponds to a small rise time or fast response.

Let, normalized bandwidth,

$$u_b = \frac{\omega_b}{\omega_n}$$

When $u=u_b$, the magnitude M, of the closed loop system is $1/\sqrt{2}$ or (-3db)

On substituting for M with $u=u_b$ and equating it to $1/\sqrt{2}$

$$M = \frac{1}{[(1-{u_b}^2)^2 + 4\zeta^2{u_b}^2]^{\frac{1}{2}}} = \frac{1}{\sqrt{2}}$$

On squaring and cross multiplying, we get,

$$(1-{u_b}^2)^2 + 4\zeta^2{u_b}^2 = 2$$

$$1 + {u_b}^4 - 2{u_b}^2 + 4\zeta^2{u_b}^2 = 2$$

$${u_b}^4 - 2{u_b}^2(1-2\zeta^2) - 1 = 0$$

Let $x = {u_b}^2$,

$$x^2 - 2x(1-2\zeta^2) - 1 = 0$$

Hence,

$$x = \frac{2(1-2\zeta^2) \pm \sqrt{4(1-2\zeta^2)^2 + 4}}{2} = \frac{2(1-2\zeta^2) \pm 2\sqrt{(1+4\zeta^4-4\zeta^2)+1}}{2}$$

Let us take only the positive sign,

$$x = 1 - 2\zeta^2 + \sqrt{(2+4\zeta^4-4\zeta^2)}$$

But, $u_b = \sqrt{x}$

$$u_b = \left[1-2\zeta^2 + \sqrt{(2+4\zeta^4-4\zeta^2)}\right]^{\frac{1}{2}}$$

Also, $u_b = \frac{\omega_b}{\omega_n}$

$$\omega_b = \omega_n \left[1 - 2\zeta^2 + \sqrt{(2 + 4\zeta^4 - 4\zeta^2)}\right]^{\frac{1}{2}}$$

Cut-off rate

The slope of the log-magnitude curve near the cut-off frequency is called cut-off rate. The cut-off rate indicates the ability of the system to distinguish the signal from noise.

Gain margin, K_g

The gain margin, K_g is defined as the value of gain, to be added to system, in order to bring the system to the verge of instability. The gain margin is given by the reciprocal of the magnitude of open loop transfer function at phase crossover frequency.

The frequency at which the phase of open loop transfer function is 180° is called the phase crossover frequency, ω_{pc}.

$$K_g = \frac{1}{|G(j\omega_{pc})|}$$

$$K_g \, in \, db = 20 \log K_g = 20 \log \frac{1}{|G(j\omega_{pc})|}$$

The gain margin in db is given by the negative of the db magnitude of G(jω) at phase crossover frequency. The gain margin indicates the additional gain that can be provided to system without affecting the stability of the system.

[Note: The gain margin of second order system is infinite].

Phase margin, γ

The phase margin, γ is defined as the additional phase lag to be added at the gain crossover frequency in order to bring the system to the verge of instability.

The gain crossover frequency, ω_{gc} is the frequency at which the magnitude of the open loop transfer function is unity (or it is the frequency at which the db magnitude is zero).

The phase margin is obtained by adding 180° to the phase angle, ϕ of the open loop transfer function at the gain crossover frequency. The phase margin indicates the additional phase lag that can be provided to the system without affecting stability.

$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

Put s=jω,

$$G(j\omega)=\frac{\omega_n^2}{j\omega(j\omega+2\zeta\omega_n)}=\frac{\omega_n^2}{\omega_n(j\frac{\omega}{\omega_n})\omega_n(j\frac{\omega}{\omega_n}+2\zeta)}=\frac{1}{(j\frac{\omega}{\omega_n})(j\frac{\omega}{\omega_n}+2\zeta)}$$

Let normalized frequency, $u=\frac{\omega}{\omega_n}$

$$G(j\omega)=\frac{1}{(ju)(ju+2\zeta)}$$

Magnitude of G(jω),

$$|G(j\omega)|=\frac{1}{(u)\sqrt{(u^2+4\zeta^2)}}=\frac{1}{\sqrt{(u^4+4u^2\zeta^2)}}$$

Phase of G(jω),

$$\angle G(j\omega)=-90^o-\tan^{-1}\frac{u}{2\zeta}$$

At the gain crossover frequency ω_{gc}, the magnitude is unity.

Hence, at $u=u_{gc}$,

$$\left|G\left(j\omega_{gc}\right)\right|=\frac{1}{\sqrt{\left({u_{gc}}^4+4{u_{gc}}^2\zeta^2\right)}}=1$$

$$\left({u_{gc}}^4+4{u_{gc}}^2\zeta^2\right)=1$$

$$\left({u_{gc}}^4+4{u_{gc}}^2\zeta^2\right)-1=0$$

Let $x={u_{gc}}^2$

$$x^2+4x\zeta^2-1=0$$

$$x=\frac{-4\zeta^2\pm\sqrt{16\zeta^4+4}}{2}=-2\zeta^2\pm\sqrt{4\zeta^4+1}$$

Let us take only the positive sign,

$$x=-2\zeta^2+\sqrt{4\zeta^4+1}$$

Hence,

$$u_{gc}=\left[-2\zeta^2+\sqrt{4\zeta^4+1}\right]^{\frac{1}{2}}$$

Phase margin,

$$\gamma=180^o+\phi_{gc}$$

$$\gamma = 180^o + \angle G(j\omega_{gc}) = 180^o + \left(-90^o - \tan^{-1}\frac{u_{gc}}{2\zeta}\right)$$

$$\gamma = 90^o - \tan^{-1}\frac{\left[-2\zeta^2 + \sqrt{4\zeta^4 + 1}\right]^{\frac{1}{2}}}{2\zeta}$$

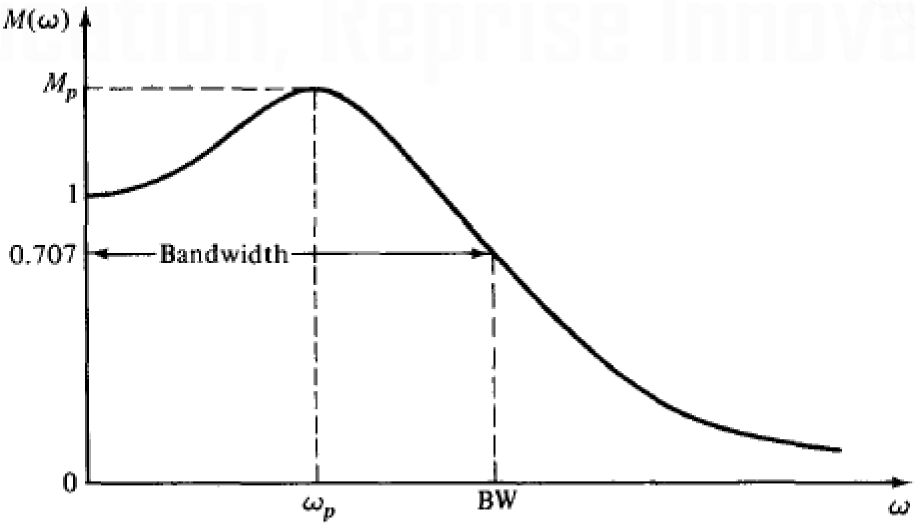

Figure 3.1.1 Typical magnification curve of a feedback control system

[Source: "Automatic Control Systems" by Benjamin C. Kuo, Page: 463]