4.2 CONTROLLABILITY AND OBSERVABILITY IN CONTROL SYSTEMS

Controllability and observability are two fundamental concepts in modern control system theory, particularly in the analysis and design of systems represented in state-space form. Controllability refers to the ability of an external input to move a system from any initial state to any desired final state within a finite period of time. In other words, a system is said to be completely controllable if every state variable can be influenced by the control input. This concept is crucial in control system design because if a system is not controllable, it is impossible to design a controller that can regulate all the internal states effectively. Mathematically, for a linear time-invariant system represented by the state equation $\dot{x} = Ax + Bu$, controllability is determined using the controllability matrix, which is formed as $[B\ AB\ A^2B\ \ldots\ A^{n-1}B]$. If this matrix has full rank equal to the number of state variables, the system is said to be completely controllable. Physically, lack of controllability implies that certain modes of the system cannot be altered regardless of the input applied, which may lead to instability or poor system performance.

On the other hand, observability is concerned with the ability to infer or reconstruct the internal states of a system based solely on its output measurements over time. A system is said to be completely observable if the current state can be uniquely determined using the output data and knowledge of the inputs. Observability plays a vital role in situations where all state variables cannot be measured directly, which is common in practical systems. In such cases, observers or estimators are designed to estimate the internal states. For a system described by $\dot{x} = Ax + Bu$ and $y = Cx + Du$, observability is evaluated using the observability matrix, which is constructed as a vertical stack of matrices $[C;\ CA;\ CA^2;\ \ldots;\ CA^{n-1}]$. If this matrix has full rank, the system is completely observable. If not, some states remain hidden and cannot be determined from the outputs.

Both controllability and observability are dual concepts and are essential for the complete analysis and design of control systems. While controllability ensures that the system can be driven as desired using inputs, observability ensures that the internal behavior of the system can be monitored and estimated accurately. These properties are prerequisites for advanced control techniques such as pole placement, state feedback control, and observer design including Luenberger observers and Kalman filters. In practical engineering applications such as robotics, aerospace systems, electrical circuits, and process control, ensuring both controllability and observability leads to better performance, stability, and reliability of the system. Thus, a thorough understanding of these concepts is indispensable for engineers working in the field of control systems.

Mathematical Condition

For a system:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

The system is observable if the **observability matrix**:

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has **full rank (n)**.

- If rank(O) = n → **Completely observable**

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- If rank(O) = n → **Completely observable** u(t) or m(t) is the control signal/manipulated signal, b(t) is the feedback signal and c(t) is the controlled output. Here, the output of the machine is fed back to a comparator(error detector).The output signal is compared with the reference input, r(t)