ROHINI COLLEGE OF ENGINEERING & TECHNOLOGY

DEPARTMENT OF MATHEMATICS

Unit IV : PRINCIPLES OF DOMINANCE

Introduction :

The principle of dominance is used to reduce the size of a games payoff matrix by eliminating a course ofaction which is so inferior to another as never to be used. Such a course of action is said to be dominated by the other. It is applicable to both pure and mixed strategy problems. However, this Step is especially useful for the evaluation of two-person zero-sum games where a saddle point does not exist.

In general, the following Steps of dominance are used to reduce the size of payoff matrix.

Step 1. If all the elements in a column are greater than or equal to the corresponding elements in another column, then that column is dominated and can be deleted from the matrix.

Step 2. If all the elements in a row are less than or equal to the corresponding elements in another row, then that row is dominated and can be deleted from the matrix.

Step 3. If all the elements in a column are greater than or equal to the average of the corresponding elements of two or more other columns, then that column can be deleted.

Step 4. If all the elements in a row are less than or equal to the average of the corresponding elements of two or more other rows, then it can be deleted.

Problem 1 Reduce the following game to 2 × 2 game using principle of dominance.

		I	II	III	IV	V	VI
	I	4	2	0	2	1	1
	II	4	3	1	3	2	2
Player *A*	III	4	3	7	–5	1	2

Player *B*	IV	4	3	4	–1	2	2
	V	4	3	3	–2	2	2

Solution: Column I, II and IV are dominated by column V, so columns I, II and VI are deleted. The reduced matrix is

Player B

		III	IV	V
Player A	I	0	2	1
	II	1	3	2
	III	7	–5	1
	IV	4	–1	2
	V	3	–2	2

Now row I is dominated by row 2 and row 5 is dominated by row 4. Hence deleting rows I and V, we have

Player B

		III	IV	V
Player A	II	1	3	2
	III	7	–5	1
	IV	4	–1	2

Now none of single row (or column) dominates another row (or column. However, column V is dominated by the average of columns III and IV. Hence deleting column V, we have

Player B

		III	IV
Player A	II	1	3
	III	7	–5
	IV	4	–1

Now average of row II and row III gives the row (4, –1) which dominates the row IV. Hence deleting row IV, we have

Player B

		III	IV
Player A	II	1	3
	III	7	–5

Problem 2

Reduce the following game into 2 × 2 game using the Steps of dominance.

		Player B		
		B_1	B_2	B_3
Player A	A_1	1	7	2
	A_2	6	2	7
	A_3	5	1	6

Solution. First, we delete the column 3 as all the elements of this column are greater than that of first column after that we delete 3rd row as all the elements of row 3 are less than the corresponding elements of row 2.

Player B

		B_1	B_2
Player A	A_1	1	7
	A_2	6	2

Hence the reduced matrix is

Mixed Strategies: Games without Saddle Point

Pure strategies are available as optimal strategies only for those games which have a saddle point. For games which do not have a saddle point can be solved by applying the concept of mixed strategies.

Algebraic Method

Consider the two-person zero-sum game with the following payoff matrix:

Player B

	Strategy	B_1	B_2	Probability
Player A	A_1	a_{11}	a_{12}	p
	A_2	a_{21}	a_{22}	$1-p$
	Probability	q	$1-q$	

If this game is to have no saddle point, the two largest elements of the matrix must constitute one of the diagonals. We have assumed this and therefore both players use mixed strategies. Our task is to determine the probabilities with which both players choose their course of action.

In this game, let player A play the strategies A_1 and A_2 with respective probabilities p and $1 - p$ and let player B play his strategies B_1 and B_2 with respective probabilities q and $1 - q$. The expected payoffs to player A when B plays any one of his strategies B_1 or B_2 throughout the game, are given by

B's Strategy	A's Strategy
B_1	$a_{11}p + a_{21}(1-p)$
B_2	$a_{12}p + a_{22}(1-p)$

Now in order that player A is unaffected with whatever choice of strategies B makes, we must have

$$a_{11}p + a_{21}(1-p) = a_{12}p + a_{22}(1-p) \qquad \Rightarrow (a_{11}-a_{12})p + (a_{22}-a_{21})p = a_{22}-a_{21}$$

$$\Rightarrow p = \frac{a_{22}-a_{21}}{(a_{11}-a_{12})+(a_{22}-a_{21})}$$

$$1\text{-}p = \frac{a_{11}-a_{12}}{(a_{11}-a_{12})+(a_{22}-a_{21})}$$

$$q = \frac{a_{22}-a_{12}}{(a_{11}-a_{21})+(a_{22}-a_{12})}$$

$$1\text{-}q = \frac{a_{11}-a_{21}}{(a_{11}-a_{21})+(a_{22}-a_{12})}$$

similarly, by equating the expected payoffs of the player *B*, for whatever choice of strategies player *A* makes, we have $a_{11}q + a_{12}(1-q) = a_{21}q + a_{22}(1-q)$

$$\Rightarrow \left[(a_{11}-a_{22})+(a_{22}-a_{12})\right]q = a_{22}-a_{12}$$

$$\text{Value of the game (V)} = \frac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}-a_{12})+(a_{22}-a_{21})}$$

(Otherwise)

From the above $a11 = a,\ a12 = b,\ a21 = c,\ a22 = d$

$$p = \frac{d-c}{(a+d)-(b+c)}$$

$$1\text{-}p = \frac{a-b}{(a+d)-(b+c)}$$

$$\text{Value of the game (V)} = \frac{ad-bc}{(a+d)-(b+c)}$$

Problem : For the following game:

Firm B

Firm A		B_1	B_2	B_3	B_4
	A_1	35	65	25	5
	A_2	30	20	15	0
	A_3	40	50	0	10
	A_4	55	60	10	15

Determine the optimal strategies for each firm and value of the game.

Solution. Since maximin value = 10 and minimax value = 15, there is no saddle point.

We apply rules of dominance to reduce the size of payoff matrix. Since each element of second row is less than the corresponding elements of first row, second row is dominated by first row. So, deleting the second row, the reduced matrix becomes

Firm B

Firm A		B_1	B_2	B_3	B_4
	A_1	35	65	25	5
	A_3	40	50	0	10
	A_4	55	60	10	15

In the reduced matrix, each element of second column is more than the corresponding elements in first column, so second column is dominated by first column. Thus after deleting the second column, the reduced matrix

$$\text{Firm } A \quad \begin{array}{c} \\ A_1 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 35 & 25 & 5 \\ 40 & 0 & 10 \\ 55 & 10 & 15 \end{bmatrix} \end{array}$$

(Firm B across the top.)

becomes

Further second row is dominated by third row, so we delete second row to get reduced matrix as

$$\begin{array}{cc} & \text{Firm } B \\ & \begin{array}{ccc} B_1 & B_3 & B_4 \end{array} \\ \text{Firm } A \begin{array}{c} A_1 \\ A_4 \end{array} & \begin{bmatrix} 35 & 25 & 5 \\ 55 & 10 & 15 \end{bmatrix} \end{array}$$

Now column one is dominated by column two. So, we delete column one and reduced matrix becomes

$$\begin{array}{cc} & \text{Firm } B \\ & \begin{array}{cc} B_3 & B_4 \end{array} \\ A_1 & \begin{bmatrix} 25 & 5 \end{bmatrix} \end{array}$$

Max {row minima} □ min {column maxima} So the game has no saddle point. Therefore it is a mixed game. We have a = 25, b = 5, c = 10 and d = 15.

$$p = \frac{d-c}{(a+d)-(b+c)} = \frac{(15-10)}{(25+15)-(5+10)} = \frac{5}{25} = 0.2$$

$$1-p = \frac{a-b}{(a+d)-(b+c)} = \frac{(25-5)}{(25+15)-(5+10)} = \frac{20}{25} = 0.8$$

$$Value\ of\ the\ Game\ (V) = \frac{ad-bc}{(a+d)-(b+c)} = \frac{(375-50)}{(25+15)-(5+10)} = \frac{325}{25} = 13$$

Problem 2: Solve the game with the following pay-off matrix:

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{cccc} I & II & III & IV \end{array} \\ \text{Player A} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 4 & 2 & 3 & 6 \\ 3 & 4 & 7 & 5 \\ 6 & 3 & 5 & 4 \end{bmatrix} \end{array}$$

Solution :

First consider the minimum of each row.

Row	Minimum Value
1	2
2	3
3	3

Maximum of {2, 3, 3} = 3

Next consider the maximum of each column.

Column	Maximum Value
1	6
2	4
3	7
4	6

Minimum of {6, 4, 7, 6}= 4

The following condition holds:

$$\text{Max \{row minima\}} \neq \text{min \{column maxima\}}$$

Therefore we see that there is no saddle point for the game under consideration. Compare columns II and III.

Column II	Column III
2	3
4	7
3	5

We see that each element in column III is greater than the corresponding element in column II. The choice is for player B. Since column II dominates column III, player B will discard his strategy 3. Now we have the reduced game

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} I & II & IV \end{array} \\ \begin{bmatrix} 4 & 2 & 6 \\ 3 & 4 & 5 \\ 6 & 3 & 4 \end{bmatrix} \end{array}$$

For this matrix again, there is no saddle point. Column II dominates column IV. The choice is for player B. So player B will give up his strategy 4

The game reduces to the following:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} I & II \end{array} \\ \begin{bmatrix} 4 & 2 \\ 3 & 4 \\ 6 & 3 \end{bmatrix} \end{array}$$

This matrix has no saddle point. The third row dominates the first row. The choice is for player A. He will give up his strategy 1 and retain strategy 3. The game reduces to the following:

$$\begin{pmatrix} 3 & 4 \\ 6 & 3 \end{pmatrix}$$

$$p = \frac{d-c}{(a+d)-(b+c)} = \frac{(3-6)}{(3+3)-(4+6)} = \frac{-3}{-4} = 0.75$$

$$1 - p = 1 - 0.75 = 0.25$$

$$Value\ of\ the\ Game\ (V) = \frac{ad-bc}{(a+d)-(b+c)} = \frac{(9-24)}{(3+3)-(4+6)} = \frac{-15}{-4} = \frac{15}{4}$$

Again, there is no saddle point. We have a 2x2 matrix. Then we have a = 3, b = 4, c = 6 and d = 3.

GRAPHICAL SOLUTION OF A 2x2 GAME WITH NO SADDLE POINT

Introduction :

The graphical method is useful for solving two person–zero–sum–game. A Game having saddle point canbe easily solved, so, we consider games without saddle point, where the payoff matrix is of size 2× n or m × 2.

Player A	B_1	B_2	B_3	Probability
$A1$	a_{11}	a_{12}	a_{13}	p_1
A_2	a_{21}	a_{22}	a_{23}	p_2
Probability	q_1	q_2	q_3	

To solve this game, we draw two vertical lines at unit distance, for representing p_1=0 and p_2=0 where p=(p_1,p_2) is the strategy of A and q= (q_1, q_2, …, $q_{n)}$ is the strategy of B. We now draw n line segments joining the points (0, a_{2j}) and (1, a_{1j}), j= 1, 2, …, n but excluding the end points. The lower envelope of these lines gives the minimum expected gain of A as a function of p_1. Thehighest point o of this lower boundary of these lines will give maximum of the minimum gain of A, i.e. maximin of A.

Problem 1:

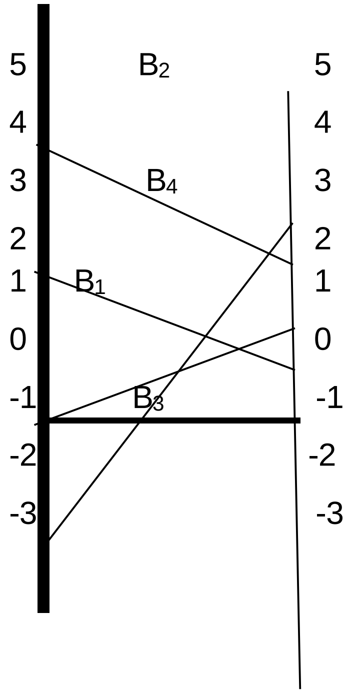
Use graphical method in solving the following game and find the value of the gameSolve the following game:

Player B

		B1	B2	B3	B4
Player A	A_1	0	5	-2	3
	A_2	2	3	4	1

Solution.

Since maximin a_{ij} = 3 < $minimax\ a_{ij}$ = 4, the game is to be solved by mixed strategies. We therefore use the graphical method to reduce this to a 2× 2 by game as follows:



We join the points 0, 5, -2 and 3 on the left line given by p_1= 0 to the points 2, 3, 4 and 1 on the right linegiven by p_2=0 respectively. Clearly the highest point of the lower envelop determines the strategies B_1 and B_4.

So, the reduced game is:

Player

B B_1

B_4

		B_1	B_4
Player A	A_1	0	3
	A_2	2	1

We have a 2x2 matrix.

Then we have a = 0, b = 3, c = 2 and d = 1.

$$p = \frac{d-c}{(a+d)-(b+c)} = \frac{(1-2)}{(0+1)-(3+2)} = \frac{-1}{-4} = 0.25$$

$$1-p = 1-0.25 = 0.75$$

$$Value\ of\ the\ Game\ (V) = \frac{ad-bc}{(a+d)-(b+c)} = \frac{(0-6)}{(0+1)-(3+2)} = \frac{-6}{-4} = \frac{3}{2}$$