

2.1 Mathematical induction:

Statement of the principle of Mathematical Induction

Let $P(n)$ be statement involving the natural number " n ".

If $P(1)$ is true.

Under the assumption that when $P(k)$ is true, $P(k+1)$ is true, then we conclude that a statement $P(n)$ is true for all natural number " n ".

Steps to prove that a statement $P(n)$ is true for all natural numbers

Step:1 We must prove that $P(1)$ is true.

Step:2 By assuming $P(k)$ is true, we must prove that $P(k+1)$ is also true.

NOTE:

Step:1 is known as the basic step.

Step:2 is known as inductive step.

Problems on Mathematical Induction:

1. Show that $1 + 2 + \dots + n = \frac{n(n+1)}{2}$ using mathematical induction.

Solution:

Let S be the set of positive integers.

To prove $p(1)$ is true.

When $n = 1$

$$\text{RHS} \Rightarrow \frac{n(n+1)}{2} = \frac{1(1+1)}{2} = 1 = \text{LHS}$$

Hence $p(1)$ is true.

Assume that $p(k)$ is true.

$$1 + 2 + \dots + k = \frac{k(k+1)}{2} \dots (1)$$

To prove $p(k+1)$ is true.

Adding $k+1$ on both sides

$$\begin{aligned} \Rightarrow 1 + 2 + \dots + k + (k+1) &= \frac{(k+1)(k+2)}{2} \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

Hence $p(k+1)$ is true.

2. Show that $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Solution:

Let S be the set of positive integers.

To prove $p(1)$ is true.

When $n = 1$

$$\text{RHS} \Rightarrow \frac{n(n+1)(2n+1)}{6} = \frac{1(1+1)(2+1)}{6} = 1 = \text{LHS}$$

Hence $p(1)$ is true.

Assume that $p(k)$ is true.

$$1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6} \quad \dots (1)$$

To prove $p(k+1)$ is true.

$$1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

Adding $(k+1)^2$ on both sides

$$\begin{aligned} \Rightarrow 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= (k+1) \left[\frac{k(2k+1)}{6} + (k+1) \right] \\ &= \frac{(k+1)}{6} [k(2k+1) + 6(k+1)] \\ &= \frac{(k+1)}{6} [2k^2 + k + 6k + 6] \end{aligned}$$

$$= \frac{(k+1)}{6} [2k^2 + 7k + 6]$$

$$= \frac{(k+1)}{6} [2k^2 + 4k + 3k + 6]$$

$$= \frac{(k+1)}{6} [2k(k+2) + 3(k+2)]$$

$$= \frac{(k+1)}{6} [(k+2) + (2k+3)]$$

Hence $p(k+1)$ is true.

3. Show that $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$

Solution:

Let S be the set of positive integers.

To prove $p(1)$ is true.

When $n = 1$

$$\text{RHS} \Rightarrow \frac{n^2(n+1)^2}{4} = \frac{1^2(1+1)^2}{4} = 1 = \text{LHS}$$

Hence $p(1)$ is true.

Assume that $p(k)$ is true.

$$1^3 + 2^3 + 3^3 + \dots + k^3 = \frac{k^2(k+1)^2}{4} \quad \dots (1)$$

To prove $p(k + 1)$ is true.

$$1^3 + 2^3 + 3^3 + \dots + k^3 + (k + 1)^3 = \frac{(k+1)^2(k+2)^2}{4}$$

Adding $(k + 1)^3$ on both sides

$$\begin{aligned} \Rightarrow 1^3 + 2^3 + 3^3 + \dots + k^3 + (k + 1)^3 &= \frac{k^2(k+1)^2}{4} + (k + 1)^3 \\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2}{4} [k^2 + 4(k + 1)] \\ &= \frac{(k+1)^2}{4} [k^2 + 4k + 4] \\ &= \frac{(k+1)^2}{4} [k^2 + 2k + 2k + 4] \\ &= \frac{(k+1)^2}{4} [k(k + 2) + 2(k + 2)] \\ &= \frac{(k+1)^2}{4} [(k + 2) + (k + 2)] \\ &= \frac{(k+1)^2(k+2)^2}{4} \end{aligned}$$

Hence $p(k + 1)$ is true.

4. Prove that $n^3 - n$ is divisible by 3, using mathematical induction .

Solution:

Let S be the set of positive integers.

To prove $p(1)$ is true.

When $n = 1$

RHS $\Rightarrow n^3 - n = 1^3 - 1 = 0$ is divisible by 3.

Hence $p(1)$ is true.

Assume that $p(k)$ is true.

$k^3 - k$ is divisible by 3.

$$\Rightarrow k^3 - k = 3m$$

$$\Rightarrow k^3 = 3m + k \dots (1)$$

To prove $p(k + 1)$ is true.

$(k + 1)^3 - (k + 1)$ is divisible by 3.

$$\Rightarrow k^3 + 1 + 3k^2 + 3k - k - 1$$

$$\Rightarrow k^3 + 3k^2 + 2k$$

$$\Rightarrow (3m + k) + 3k^2 + 2k$$

$$\Rightarrow 3m + 3k^2 + 3k$$

$$\Rightarrow 3(m + k^2 + k) \text{ is divisible by 3.}$$

Hence $p(k + 1)$ is true.

5. Prove that $8^n - 3^n$ is a multiple of 5. .

Solution:

Let S be the set of positive integers.

To prove $p(1)$ is true.

When $n = 1$

RHS $\Rightarrow 8^n - 3^n = 8^1 - 3^1 = 5$ is a multiple of 5 which is true.

Hence $p(1)$ is true.

Assume that $p(k)$ is true.

$8^k - 3^k$ is a multiple of 5.

$$\Rightarrow 8^k - 3^k = 5m$$

$$\Rightarrow 8^k = 5m + 3^k \dots (1)$$

To prove $p(k + 1)$ is true.

$8^{k+1} - 3^{k+1}$ is a multiple of 5.

$$\Rightarrow 8 \cdot 8^k - 3 \cdot 3^k$$

$$\Rightarrow (5m + 3^k) \cdot 8 - 3 \cdot 3^k$$

$$\Rightarrow 5 \cdot 8m + 8 \cdot 3^k - 3 \cdot 3k$$

$$\Rightarrow 5 \cdot 8m + 5 \cdot 3^k$$

$$\Rightarrow 5(8m + 3^k) \text{ is a multiple of 5.}$$

Hence $p(k + 1)$ is true.

6. State and prove Handshaking theorem.

Suppose there are “ n ” people in a room, $n \geq 1$ and that they all shake hands with one another, prove that $\frac{n(n-1)}{2}$ handshakes will have accrued.

Solution:

Let S be the set of positive integers.

To prove $p(1)$ is true. ★

When $n = 1$

$$p(1) = \frac{n(n-1)}{2} = \frac{1(1-1)}{2} = 0$$

\Rightarrow there is no handshake accrued which means there is only one person.

Hence $p(1)$ is true.

Assume that $p(k)$ is true.

$$p(k) = \frac{k(k-1)}{2} \quad \dots (1)$$

To prove $p(k + 1)$ is true.

$$p(k + 1) = \frac{(k+1)k}{2}$$

Suppose if one person entered into the room then he will shake his hand with “k” other person whenever $p(k)$ is true.

Hence $p(k + 1)$ is true by mathematical induction.

The Well – Ordering Property:

The validity of mathematical induction follows from the following fundamental axioms about the set of integers.

Every non – empty set of non – negative integers has a least element.

The well ordering property can often be used directly in the proof.

