Morphological Image Processing

1. Introduction

Morphology is a broad set of image-processing operations that process an image based on shapes, using a structuring element (SE) — a small matrix/set B applied to an input image A. Morphological operations extract image components useful for representing and describing region shape, such as boundaries, skeletons, and the convex hull.

Morphological operations were originally defined for binary (black-and-white) images and later extended to gray-scale images. They are based on set theory concepts — union, intersection, complement, translation, and reflection of sets.

Basic Set Definitions:

- Translation of A by point z: $(A)_z = \{c \mid c = a + z, \text{ for } a \in A\}$
- Reflection of B: $\hat{B} = \{w \mid w = -b, \text{ for } b \in B\}$
- Complement of A: $A^c = \{w \mid w \notin A\}$

2. Erosion

Erosion shrinks or thins objects in a binary image. The erosion of A by structuring element B is defined as:

$$A \ominus B = \{z \mid (B)_z \subseteq A\}$$

The erosion of A by B is the set of all points z such that B, translated by z, is completely contained in A. It is used to remove small noise elements, detach thinly connected objects, and shrink object boundaries.

3. Dilation

Dilation grows or thickens objects in a binary image. The dilation of A by structuring element B is defined as:

$$A \oplus B = \{z \mid (\hat{B})_z \cap A \neq \emptyset\}$$

The dilation of A by B is the set of all displacements z such that $\hat{B}$ (the reflection of B) and A overlap by at least one element. Dilation is used to fill small holes and gaps, and to bridge small breaks in an object.

Duality: Erosion and dilation are duals of each other with respect to complementation and reflection:

$$(A \ominus B)^c = A^c \oplus \hat{B}$$

4. Opening and Closing

(a) Opening

Opening smooths the contour of an object, breaks narrow isthmuses, and eliminates thin protrusions. It is erosion followed by dilation:

$$A \circ B = (A \ominus B) \oplus B$$

Geometric interpretation: the union of all translations of B that fit entirely within A (the SE 'rolls' inside the object boundary).

(b) Closing

Closing also smooths contours, but generally fuses narrow breaks, fills long thin gulfs, and eliminates small holes. It is dilation followed by erosion:

$$A \bullet B = (A \oplus B) \ominus B$$

Duality: $(A \bullet B)^c = (A^c \circ \hat{B})$

Properties:

- Opening and closing are idempotent: $(A \circ B) \circ B = A \circ B$ and $(A \bullet B) \bullet B = A \bullet B$
- $A \circ B$ is a subset of A; A is a subset of $A \bullet B$

5. The Hit-or-Miss Transform

A basic tool for shape detection. Uses two disjoint structuring elements B_1 (object) and B_2 (background):

$$A \circledast B = (A \ominus B_1) \cap (A^c \ominus B_2)$$

It locates all pixel positions where the shape of B_1 matches the foreground exactly and B_2 matches the background exactly — used to detect specific shapes/patterns and object corners.

6. Basic Morphological Algorithms

(a) Boundary Extraction

$$\beta(A) = A - (A \ominus B) = A \cap (A \ominus B)^c$$

The boundary is obtained by eroding A by B, then subtracting the eroded image from A.

(b) Region (Hole) Filling

Starting from a point p inside a boundary-defined region, dilation is repeated (constrained by the complement of A) until it stabilizes:

$$X_k = (X_{k-1} \oplus B) \cap A^c, k = 1, 2, 3\ldots$$

Iteration stops when $X_k = X_{k-1}$; the filled region is $X_k \cup A$.

(c) Extraction of Connected Components

$$X_k = (X_{k-1} \oplus B) \cap A, k = 1, 2, 3\ldots$$

Starting from a seed point p in a connected component, repeated dilation constrained by A extracts the entire connected component.

(d) Convex Hull

The smallest convex set containing A, computed iteratively using four structuring elements (one per quadrant direction) applied with the hit-or-miss transform until convergence.

(e) Thinning

$$A \otimes B = A - (A \circledast B) = A \cap (A \circledast B)^c$$

Thinning reduces objects to thin (skeletal) strokes, typically one pixel wide, using a sequence of structuring elements applied successively.

(f) Thickening

$$A \odot B = A \cup (A \circledast B)$$

The morphological dual of thinning; used to grow selected regions of a binary image (usually applied to the background and then complemented).

(g) Skeletons

The skeleton S(A) of a set A can be expressed in terms of erosions and openings:

$$S(A) = \bigcup_{k=0}^{K} S_k(A), \text{ where } S_k(A) = (A \ominus kB) - [(A \ominus kB) \circ B]$$

Here $(A \ominus kB)$ means k successive erosions of A by B, and K is the last iterative step before A erodes to an empty set.

(h) Pruning

Pruning removes parasitic (spur) branches produced by skeletonization or thinning algorithms — typically implemented as a thinning pass with end-point structuring elements, followed by conditional dilation to restore the main structure.

7. Gray-Scale Morphology

Erosion and dilation extend to gray-scale images f(x,y) with structuring element b(x,y):

$$(f \ominus b)(x,y) = \min\{f(x+s, y+t) - b(s,t)\}$$

$$(f \oplus b)(x,y) = \max\{f(x-s, y-t) + b(s,t)\}$$

Gray-scale erosion darkens the image (min filter) and shrinks bright regions; gray-scale dilation brightens the image (max filter) and grows bright regions. Opening and closing are defined analogously and are used for applications such as:

- Morphological smoothing (opening followed by closing)
- Morphological gradient: $g = (f \oplus b) - (f \ominus b)$, highlights sharp intensity transitions (edges)
- Top-hat transformation: $T_{hat}(f) = f - (f \circ b)$, used for shading correction / extracting small bright details
- Bottom-hat transformation: $T_{eat}(f) = (f \bullet b) - f$, extracts small dark details
- Granulometry — determining size distribution of particles using openings of increasing size

8. Summary Table

Operation	Definition (Set Notation)	Effect
Erosion	$A \ominus B = \{z \mid (B)_z \subseteq A\}$	Shrinks objects, removes small protrusions
Dilation	$A \oplus B = \{z \mid (\hat{B})_z \cap A \neq \emptyset\}$	Grows objects, fills small holes/gaps
Opening	$A \circ B = (A \ominus B) \oplus B$	Smooths contours, removes small objects/spikes
Closing	$A \bullet B = (A \oplus B) \ominus B$	Fills small holes and gaps, smooths contours
Hit-or-Miss	$A \square B = (A \square B_1) \cap (A^c \ominus B_2)$	Shape/pattern detection

9. Applications

- Noise removal (opening/closing)
- Edge and boundary detection
- Region filling and object counting

- Skeletonization for shape analysis and pattern recognition
- Text/character segmentation in document image processing
- Medical image analysis (cell/tissue shape extraction)

10. Conclusion

Morphological image processing provides a powerful, mathematically rigorous set of tools — rooted in set theory — for analyzing and manipulating the geometric structure of objects within an image. Fundamental operators (erosion, dilation, opening, closing, hit-or-miss) combine to build higher-level algorithms (boundary extraction, region filling, thinning, skeletonization, pruning) that are indispensable in shape analysis, pre-processing, and feature extraction across binary and gray-scale image processing applications.