

Contour Integration

Evaluation of Real Integrals

The evaluation of certain types of real definite integrals of complex functions over suitable closed paths or contours and applying Cauchy's Residue theorem is known as Contour Integration.

Type 1: Integration round the unit circle

Integrals of the form $\int_0^{2\pi} f(\cos \theta, \sin \theta) d\theta$ where f is a rational function in $\cos \theta$ and $\sin \theta$

To evaluate this type of integrals

We take the unit circle $|z| = 1$ as the contour C .

On $|z| = 1$, let $z = e^{i\theta}$

$$\Rightarrow \frac{dz}{d\theta} = ie^{i\theta} = iz$$

$$\therefore d\theta = \frac{dz}{iz}$$

$$\text{Also, } \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2} = \frac{z^2 + 1}{2z}$$

$$\text{and, } \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z^2 - 1}{2iz}$$

$$|z| = 1 \Rightarrow \theta \text{ varies from } 0 \text{ to } 2\pi$$

$$\therefore \int_0^{2\pi} f(\cos \theta, \sin \theta) d\theta = \int_C f\left(\frac{z^2 + 1}{2z}, \frac{z^2 - 1}{2iz}\right) \frac{dz}{iz}$$

Now applying Cauchy's Residue theorem, we can evaluate the integral on the right side.

Problems based on Contour Integration

Example: 4.57 Evaluate $\int_0^{2\pi} \frac{d\theta}{5 + 4 \sin \theta}$ using Contour integration.

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz} \text{ and } \sin \theta = \frac{z^2 - 1}{2iz}$$

$$\begin{aligned} \therefore \int_0^{2\pi} \frac{d\theta}{5 + 4 \sin \theta} &= \int_C \frac{\frac{dz}{iz}}{5 + 4\left(\frac{z^2 - 1}{2iz}\right)} \text{ where } C \text{ is } |z| = 1 \\ &= \int_C \frac{\frac{dz}{iz}}{\frac{5iz + 2z^2 - 2}{iz}} \\ &= \int_C \frac{dz}{2z^2 + 5iz - 2} \\ &= \int_C f(z) dz \quad \dots (1) \end{aligned}$$

$$\text{Where, } f(z) = \frac{1}{2z^2 + 5iz - 2}$$

To Evaluate, $\int_C f(z) dz$

To find poles of $f(z)$, put $2z^2 + 5iz - 2 = 0$

$$z = \frac{-5i \pm \sqrt{-25 + 16}}{4} = \frac{-5i \pm 3i}{4}$$

$$z = -\frac{i}{2}, -2i \text{ are poles of order one}$$

Given C is $|z| = 1$

Consider $z = -\frac{i}{2}$

$$\Rightarrow |z| = \left| -\frac{i}{2} \right| = \frac{1}{2} < 1$$

$\therefore z = -\frac{i}{2}$ lies inside C

Consider $z = -2i$

$$\Rightarrow |z| = |-2i| = 2 > 1$$

$\therefore z = -2i$ lies outside C .

Find the residue for inside pole $z = -\frac{i}{2}$

$$\begin{aligned} [\text{Res } f(z)]_{z=-\frac{i}{2}} &= \lim_{z \rightarrow -\frac{i}{2}} \left(z + \frac{i}{2} \right) f(z) \\ &= \lim_{z \rightarrow -\frac{i}{2}} \left(z + \frac{i}{2} \right) \frac{1}{z \left(z + \frac{i}{2} \right) (z + 2i)} \\ &= \frac{1}{z \left(-\frac{i}{2} + 2i \right)} = \frac{1}{3i} \end{aligned}$$

$\therefore$ By Cauchy's residue theorem

$$\int_C f(z) dz = 2\pi i \quad [\text{Sum of residues}]$$

$$= 2\pi i \left(\frac{1}{3i} \right) = \frac{2\pi}{3}$$

$$(1) \Rightarrow \int_0^{2\pi} \frac{d\theta}{5+4 \sin \theta} = \frac{2\pi}{3}$$

Example: 4.58 Evaluate $\int_0^{2\pi} \frac{d\theta}{13+5 \sin \theta}$ using Contour Integration.

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz} \text{ and } \sin \theta = \frac{z^2-1}{2iz}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{13+5 \sin \theta} = \int_C \frac{dz/iz}{13+5 \left(\frac{z^2-1}{2iz} \right)} \text{ where } C \text{ is } |z| = 1$$

$$= \int_C \frac{dz/iz}{\frac{26iz+5z^2-5}{2iz}}$$

$$= 2 \int_C \frac{dz}{5z^2+26iz-5}$$

$$= 2 \int_C f(z) dz \quad \dots (1)$$

$$\text{Where, } f(z) = \frac{1}{5z^2+26iz-5}$$

To evaluate $\int_C f(z) dz$

To find poles of $f(z)$, put $5z^2 + 26iz - 5 = 0$

$$z = \frac{-26i \pm \sqrt{-676+100}}{10} = \frac{-26i \pm 24i}{10}$$

$$\Rightarrow z = -\frac{i}{5}, -5i \text{ are poles of order one.}$$

Given C is $|z| = 1$

Consider $z = -\frac{i}{5}$

$$\Rightarrow |z| = \left| -\frac{i}{5} \right| = \frac{1}{5} < 1$$

$\therefore z = -\frac{i}{5}$ lies inside C

Consider $z = -5i$

$$\Rightarrow |z| = |-5i| = 5 > 1$$

$\therefore z = -5i$ lies outside C.

Find the residue for inside pole $z = -\frac{i}{5}$

$$\begin{aligned} [\text{Res } f(z)]_{z=-\frac{i}{5}} &= \lim_{z \rightarrow -\frac{i}{5}} \left(z + \frac{i}{5} \right) f(z) \\ &= \lim_{z \rightarrow -\frac{i}{5}} \left(z + \frac{i}{5} \right) \frac{1}{5z^2 + 26iz - 5} \\ &= \lim_{z \rightarrow -\frac{i}{5}} \left(z + \frac{i}{5} \right) \frac{1}{(5z+i)(z+5i)} \\ &= \lim_{z \rightarrow -\frac{i}{5}} \left(z + \frac{i}{5} \right) \frac{1}{(5(z+\frac{i}{5}))(z+5i)} \\ &= \frac{1}{5(-\frac{i}{5}+5i)} = \frac{1}{24i} \end{aligned}$$

$\therefore$ By Cauchy's residue theorem

$$\begin{aligned} \int_C f(z) dz &= 2\pi i [\text{Sum of residues}] \\ &= 2\pi i \left(\frac{1}{24i} \right) = \frac{\pi}{12} \\ \Rightarrow \int_0^{2\pi} \frac{d\theta}{13+5 \sin \theta} &= 2 \left(\frac{\pi}{12} \right) = \frac{\pi}{6} \end{aligned}$$

Example: 4.59 Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b \cos \theta}$, $a > b > 0$ by using contour integration.

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz} \text{ and } \cos \theta = \frac{z^2+1}{2z}$$

$$\begin{aligned} \therefore \int_0^{2\pi} \frac{d\theta}{a+b \cos \theta} &= \int_C \frac{dz/iz}{a+b \left(\frac{z^2+1}{2z} \right)} \text{ where } C \text{ is } |z| = 1 \\ &= \int_C \frac{dz/iz}{\frac{2az+bz^2+b}{2z}} \\ &= \frac{2}{i} \int_C \frac{dz}{bz^2+2az+b} \\ &= \frac{2}{i} \int_C f(z) dz \quad \dots (1) \end{aligned}$$

Where, $f(z) = \frac{1}{bz^2+2az+b}$

To evaluate $\int_C f(z) dz$

To find poles of $f(z)$, put $bz^2 + 2az + b$

$$z = \frac{-2a \pm \sqrt{4(a^2-b^2)}}{2b} = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$$z = \frac{-a+\sqrt{a^2-b^2}}{b}, \frac{-a-\sqrt{a^2-b^2}}{b} \text{ are poles of order one.}$$

Clearly, $z = \frac{-a+\sqrt{a^2-b^2}}{b} = \alpha$ lies inside c

and $z = \frac{-a-\sqrt{a^2-b^2}}{b} = \beta$ lies outside c

Since $a > b$, we can write $bz^2 + 2az + b = b(z - \alpha)(z - \beta)$

Find the residue for inside pole $z = \alpha$

$$\begin{aligned} \text{Res } f(z)]_{z=\alpha} &= \lim_{z \rightarrow \alpha} (z - \alpha) \frac{1}{b(z - \alpha)(z - \beta)} \\ &= \frac{1}{b(\alpha - \beta)} \\ &= \frac{1}{2\sqrt{a^2 - b^2}} \end{aligned}$$

$\therefore$ By Cauchy residue theorem

$$\begin{aligned} \int_c f(z) dz &= 2\pi i [\text{sum of residues}] \\ &= 2\pi i \left[\frac{1}{2\sqrt{a^2 - b^2}} \right] \\ &= \frac{\pi i}{\sqrt{a^2 - b^2}} \\ (1) \Rightarrow \int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} &= \frac{2}{i} \left[\frac{\pi i}{\sqrt{a^2 - b^2}} \right] \\ &= \frac{2\pi}{\sqrt{a^2 - b^2}} \end{aligned}$$

Example: 4.60 Show that $\int_0^{2\pi} \frac{d\theta}{1 - 2p \cos \theta + p^2} = \frac{2\pi}{1 - p^2}$ if $|p| < 1$

Solution:

Replacement Let $z = e^{i\theta}$

$$\begin{aligned} \Rightarrow d\theta &= \frac{dz}{iz} \text{ and } \cos \theta = \frac{z^2 + 1}{2z} \\ \therefore \int_0^{2\pi} \frac{d\theta}{1 - 2p \cos \theta + p^2} &= \int_c \frac{dz/iz}{\left[1 - 2p \left(\frac{z^2 + 1}{2z} \right) + p^2 \right]} \end{aligned}$$

Where, C is $|z| = 1$

$$\begin{aligned} &= \int_c \frac{dz}{iz \left[\frac{z - P(z^2 + 1) + P^2 z}{z} \right]} \\ &= -\frac{1}{i} \int_c \frac{dz}{P(z^2 + 1) - P^2 z - z} \\ &= -\frac{1}{i} \int_c \frac{dz}{P(z^2 - (1 + P^2)z + P)} \\ &= -\frac{1}{i} \int_c f(z) dz \quad \dots (1) \end{aligned}$$

Where, $f(z) = \frac{1}{Pz^2 - (1 + P^2)z + P}$

To evaluate $\int_c f(z) dz$

To find poles of $f(z)$, put $Pz^2 - (1 + P^2)z + P = 0$

$$z = \frac{(P^2 + 1) \pm \sqrt{(P^2 + 1)^2 - 4P^2}}{2P}$$

$$= \frac{(P^2+1)\pm(P^2-1)}{2p}$$

$\Rightarrow z = p, \frac{1}{p}$ are poles of order one.

Since $|p| < 1$, the pole $z = p$ lies inside C and the pole $z = \frac{1}{p}$ lies outside C.

Find the residue for the inside pole $z = p$

$$\begin{aligned} [Res f(z)]_{z=p} &= \lim_{z \rightarrow p} (z - p) f(z) \\ &= \lim_{z \rightarrow p} \left[(z - p) \frac{1}{p(z-p)\left(z-\frac{1}{p}\right)} \right] \\ &= \frac{1}{p\left(p-\frac{1}{p}\right)} = \frac{1}{p^2-1} \end{aligned}$$

$\therefore$ By Cauchy Residue theorem

$$\begin{aligned} \int_C f(z) dz &= 2\pi i [\text{sum of residues}] \\ &= 2\pi i \left[\frac{1}{p^2-1} \right] \\ (1) \Rightarrow \int_0^{2\pi} \frac{d\theta}{1-2p \cos \theta + p^2} &= -\frac{1}{i} \left[\frac{2\pi i}{p^2-1} \right] \\ &= \frac{2\pi}{1-p^2} \end{aligned}$$

Example: 4.61 Evaluate $\int_0^{2\pi} \frac{\cos 3\theta d\theta}{5-4 \cos \theta}$ using contour integration.

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz} \text{ and } \cos \theta = \frac{z^2+1}{2z}$$

$$\cos 3\theta = \text{Real part of } e^{i3\theta} = R.P (z^3)$$

$$\begin{aligned} \therefore \int_0^{2\pi} \frac{\cos 3\theta d\theta}{5-4 \cos \theta} &= \int_0^{2\pi} \frac{R.P \left(\frac{z^3 dz}{iz} \right)}{5-4 \left(\frac{z^2+1}{2z} \right)} \text{ where } C \text{ is } |z| = 1 \\ &= R.P \int_C \frac{z^3 dz / iz}{\frac{5z - (2z^2+2)}{z}} \\ &= R.P \left(-\frac{1}{i} \right) \int_C f(z) dz \quad \dots (1) \end{aligned}$$

$$\text{Where, } f(z) = \frac{z^3}{2z^2-5z+2}$$

To evaluate $\int_C f(z) dz$

To find poles of $f(z)$, put $2z^2 - 5z + 2 = 0$

$$z = \frac{5 \pm \sqrt{25-16}}{4} = \frac{5 \pm 3}{4}$$

$\Rightarrow z = 2, \frac{1}{2}$ are poles of order one.

Given C is $|z| = 1$

Consider $z = \frac{1}{2}$

$$\Rightarrow |z| = \left| \frac{1}{2} \right| = \frac{1}{2} < 1$$

$\therefore z = \frac{1}{2}$ lies inside C

Consider $z = 2$

$$\Rightarrow |z| = |2| = 2 > 1$$

$\therefore z = 2$ lies outside C

Find the residue for inside pole $z = \frac{1}{2}$

$$\begin{aligned} [Res f(z)]_{z=\frac{1}{2}} &= \lim_{z \rightarrow \frac{1}{2}} \left(z - \frac{1}{2} \right) f(z) \\ &= \lim_{z \rightarrow \frac{1}{2}} \left(z - \frac{1}{2} \right) f(z) \\ &= \lim_{z \rightarrow \frac{1}{2}} \left(z - \frac{1}{2} \right) \frac{z^3}{(2z-1)(z-2)} \\ &= \lim_{z \rightarrow \frac{1}{2}} \left(z - \frac{1}{2} \right) \frac{z^3}{2\left(z-\frac{1}{2}\right)(z-2)} \\ &= \frac{\left(\frac{1}{2}\right)^3}{2\left(\frac{1}{2}-2\right)} = -\frac{1}{24} \end{aligned}$$

$\therefore$ By Cauchy's Residue theorem

$$\begin{aligned} \int_C f(z) dz &= 2\pi i [\text{sum of residues}] \\ &= 2\pi i \left(-\frac{1}{24} \right) = -\frac{\pi i}{12} \end{aligned}$$

$$(1) \Rightarrow \int_0^{2\pi} \frac{\cos 3\theta}{5-4 \cos \theta} d\theta = R.P \left(-\frac{1}{i} \right) \left(-\frac{\pi i}{12} \right) = \frac{\pi}{12}$$

Example: 4.62 Evaluate $\int_0^{2\pi} \frac{\sin^2 \theta d\theta}{5-3 \cos \theta} = \int_0^{2\pi} \frac{\cos 2\theta d\theta}{10-6 \cos \theta}$

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz} \text{ and } \cos \theta = \frac{z^2+1}{2z}$$

$$\cos 2\theta = \text{Real part of } e^{i2\theta} = R.P (Z^2)$$

$$\therefore \int_0^{2\pi} \frac{\sin^2 \theta}{5-3 \cos \theta} d\theta = \int_0 \frac{1-R.P \left(\frac{z^2}{iz} \right) dz}{10-6 \left(\frac{z^2+1}{2z} \right)}$$

where C is $|z| = 1$

$$\begin{aligned} &= R.P \int_C \frac{(1-z^2) dz / iz}{\frac{10z-3z^2-3}{z}} \\ &= R.P \left(-\frac{1}{i} \right) \int_C \frac{(1-z^2) dz}{3z^2-10z+3} \\ &= R.P \left(-\frac{1}{i} \right) \int_C f(z) dz \quad \dots (1) \end{aligned}$$

$$\text{Where, } f(z) = \frac{1-z^2}{3z^2-10z+3}$$

To evaluate $\int_C f(z) dz$

To find poles of $f(z)$, put $3z^2 - 10z + 3 = 0$

$$z = \frac{10 \pm \sqrt{100-36}}{6} = \frac{10 \pm 8}{6}$$

$\therefore z = 3, \frac{1}{3}$ are poles of order one.

Given C is $|z| = 1$

Consider $z = \frac{1}{3}$

$$\Rightarrow |z| = \left| \frac{1}{3} \right| = \frac{1}{3} < 1$$

$\therefore z = \frac{1}{3}$ lies inside C

Consider $z = 3$

$$\Rightarrow |z| = |3| = 3 > 1$$

$\therefore z = 3$ lies outside C

Find the residue for inside pole $z = \frac{1}{3}$

$$\begin{aligned} [Res f(z)]_{z=\frac{1}{3}} &= \lim_{z \rightarrow \frac{1}{3}} \left(z - \frac{1}{3} \right) f(z) \\ &= \lim_{z \rightarrow \frac{1}{3}} \left(z - \frac{1}{3} \right) \frac{(1-z^2)}{3 \left(z - \frac{1}{3} \right) (z-3)} \\ &= \frac{1 - \left(\frac{1}{3} \right)^2}{3 \left(\frac{1}{3} - 3 \right)} = -\frac{1}{9} \end{aligned}$$

$\therefore$ By Cauchy's Residue theorem

$$\begin{aligned} \int_C f(z) dz &= 2\pi i [\text{sum of residues}] \\ &= 2\pi i \left(-\frac{1}{9} \right) \end{aligned}$$

$$(1) \Rightarrow \int_0^{2\pi} \frac{\sin^2 \theta d\theta}{5-3 \cos \theta} = R.P \left(-\frac{1}{i} \right) \left(-\frac{2\pi i}{9} \right) = \frac{2\pi}{9}$$

Example: 4.63 Using Contour Integration, evaluate the real integral $\int_0^\pi \frac{1+2 \cos \theta}{5+4 \cos \theta} d\theta$

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz} \text{ and } \cos \theta = \frac{z^2+1}{2z}$$

$$\text{Now, } \int_0^\pi \frac{1+2 \cos \theta}{5+4 \cos \theta} d\theta = \frac{1}{2} \int_0^{2\pi} \frac{1+2 \cos \theta}{5+4 \cos \theta} d\theta$$

$$\left[\because \int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx, \text{ if } f(2a-x) = f(x) \right]$$

$$\begin{aligned} \therefore \frac{1}{2} \int_0^{2\pi} \frac{1+2 \cos \theta}{5+4 \cos \theta} d\theta &= \frac{1}{2} \int_C \frac{1 + \left[1 + 2 \left(\frac{z^2+1}{2z} \right) \right] \frac{dz}{iz}}{5 + 4 \left(\frac{z^2+1}{2z} \right)} \\ &= \frac{1}{2i} \int_C \frac{(z^2+z+1)}{z(2z^2+5z+2)} \\ &= \frac{1}{2i} \int_C f(z) dz \quad \dots (1) \end{aligned}$$

$$\text{Where, } f(z) = \frac{z^2+z+1}{z(2z^2+5z+2)}$$

To evaluate $\int_C f(z) dz$

To find poles of $f(z)$, put $z(2z^2 + 5z + 2) = 0$

$$\Rightarrow z = 0; 2z^2 + 5z + 2 = 0$$

$$\Rightarrow z = 0; z = -2, z = -\frac{1}{2} \text{ are poles of order one.}$$

Given C is $|z| = 1$

Consider $z = 0$

$$\Rightarrow |z| = |0| = 0 < 1$$

$\therefore z = 0$ lies inside C

Consider $z = -2$

$$\Rightarrow |z| = |-2| = 2 > 1$$

$\therefore z = -2$ lies outside C

Consider $z = -\frac{1}{2}$

$$\Rightarrow |z| = \left| -\frac{1}{2} \right| = \frac{1}{2} < 1$$

$\therefore z = -\frac{1}{2}$ lies inside C

Find the residue for the inside pole

(i) When $z = 0$

$$[Res f(z)]_{z=0} = \lim_{z \rightarrow 0} (z - 0)f(z)$$

$$\lim_{z \rightarrow 0} z \frac{(z^2 + z + 1)}{z(2z^2 + 5z + 2)} = \frac{1}{2}$$

(ii) When $z = -\frac{1}{2}$

$$\begin{aligned} [Res f(z)]_{z=-\frac{1}{2}} &= \lim_{z \rightarrow -\frac{1}{2}} \left(z + \frac{1}{2} \right) f(z) \\ &= \lim_{z \rightarrow -\frac{1}{2}} \left(z + \frac{1}{2} \right) \frac{z^2 + z + 1}{z(2z + 1)(z + 2)} \\ &= \lim_{z \rightarrow -\frac{1}{2}} \left(z + \frac{1}{2} \right) \frac{z^2 + z + 1}{z \cdot 2 \left(z + \frac{1}{2} \right) (z + 2)} \\ &= \frac{\frac{1}{4} - \frac{1}{2} + 1}{2 \left(-\frac{1}{2} \right) \left(-\frac{1}{2} + 2 \right)} \\ &= \frac{\frac{3}{4}}{-\frac{3}{2}} = -\frac{1}{2} \end{aligned}$$

$\therefore$ By Cauchy's Residue Theorem

$$\begin{aligned} \int_C f(z) dz &= 2\pi i [\text{Sum of residues}] \\ &= 2\pi i \left[\frac{1}{2} - \frac{1}{2} \right] = 0 \end{aligned}$$

$$(1) \Rightarrow \frac{1}{2} \int_0^{2\pi} \frac{1 + 2 \cos \theta}{5 + 4 \cos \theta} d\theta = \frac{1}{2i} [0] = 0$$

Example: 4.64 Evaluate $\int_0^{2\pi} \frac{d\theta}{2 + \cos \theta}$ using Contour Integration.

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz} \text{ and } \cos \theta = \frac{z^2+1}{2z}$$

$$\begin{aligned} \therefore \int_0^{2\pi} \frac{d\theta}{2+\cos \theta} &= \int_C \frac{dz/iz}{2+\left(\frac{z^2+1}{2z}\right)} \text{ where } C \text{ is } |z| = 1 \\ &= \int_C \frac{dz/iz}{\frac{z^2+4z+1}{2z}} \\ &= \frac{2}{i} \int_C \frac{dz}{z^2+4z+1} \\ &= \frac{2}{i} \int_C f(z)dz \quad \dots (1) \end{aligned}$$

$$\text{Where, } f(z) = \frac{1}{z^2+4z+1}$$

To evaluate $\int_C f(z)dz$

To find poles of $f(z)$, put $z^2 + 4z + 1 = 0$

$$z = \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm 2\sqrt{3}}{2}$$

$\therefore z = -2 + \sqrt{3}, -2 - \sqrt{3}$ are the poles of order one.

Given C is $|z| = 1$

Consider $z = -2 + \sqrt{3}$

$$\Rightarrow |z| = |-2 + \sqrt{3}| < 1$$

$\therefore z = -2 + \sqrt{3}$ lies inside C

Consider $z = -2 - \sqrt{3}$

$$\Rightarrow |z| = |-2 - \sqrt{3}| > 1$$

$\therefore z = -2 - \sqrt{3}$ lies outside C

Find the residue for the inside pole $z = -2 + \sqrt{3}$

$$\begin{aligned} [Res f(z)]_{z=-2+\sqrt{3}} &= \lim_{z \rightarrow -2+\sqrt{3}} (z - (-2 + \sqrt{3}))f(z) \\ \lim_{z \rightarrow -2+\sqrt{3}} [z - (-2 + \sqrt{3})] \frac{1}{[z - (-2 + \sqrt{3})][z - (-2 - \sqrt{3})]} \\ &= \frac{1}{-2 + \sqrt{3} + 2 + \sqrt{3}} = \frac{1}{2\sqrt{3}} \end{aligned}$$

$\therefore$ By Cauchy's Residue Theorem

$$\begin{aligned} \int_C f(z)dz &= 2\pi i [\text{Sum of residues}] \\ &= 2\pi i \left[\frac{1}{2\sqrt{3}} \right] = \frac{\pi i}{\sqrt{3}} \end{aligned}$$

$$(1) \Rightarrow \int_0^{2\pi} \frac{d\theta}{2+\cos \theta} = \frac{2}{i} \left[\frac{\pi i}{\sqrt{3}} \right] = \frac{2\pi}{\sqrt{3}}$$

Exercise: 4.4

Evaluate the following Integrals

$$1) \int_0^{2\pi} \frac{d\theta}{1+2\cos \theta} \quad \text{Ans: } \frac{2\pi}{\sqrt{3}}$$

$$2) \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos \theta} d\theta \quad \text{Ans: } \frac{\pi}{6}$$

$$3) \int_0^{2\pi} \frac{\sin^2 \theta}{5+4\cos \theta} d\theta \quad \text{Ans: } \frac{\pi}{4}$$

$$\begin{array}{ll}
4) \int_0^{2\pi} \frac{d\theta}{5-4 \cos \theta} & \text{Ans: } \frac{2\pi}{3} \\
5) \int_0^{2\pi} \frac{\cos 2\theta}{1-2a \cos \theta + a^2}; |a| < 1 & \text{Ans: } \frac{2\pi a^3}{1-a^2} \\
6) \int_0^{2\pi} \frac{d\theta}{1-2x \sin \theta + x^2}; [x < n < 1] & \text{Ans: } \frac{2\pi}{1-x^2} \\
7) \int_0^{2\pi} \frac{d\theta}{17-8 \cos \theta} & \text{Ans: } \frac{2\pi}{15} \\
8) \int_0^{2\pi} \frac{d\theta}{13+12 \cos \theta} & \text{Ans: } \frac{2\pi}{5} \\
9) \int_0^{2\pi} \frac{d\theta}{13+12 \cos \theta} & \text{Ans: } \frac{2\pi}{b^2} \left[a - \sqrt{a^2 - b^2} \right] \\
10) \int_0^{2\pi} \frac{\cos^2 3\theta}{5-4 \cos \theta} d\theta & \text{Ans: } \frac{3\pi}{8}
\end{array}$$

Type II: Integration around semi – circular contour

Integrals of the form $\int_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx$,

where $f(x)$ and $g(x)$ are polynomials in x , such that the degree of $f(x)$ is less than that of $g(x)$ atleast by two and $g(x)$ does not vanish for any value of x .

Let C be a closed contour of real axis from $-R$ to R and semicircle ' S ' of radius R above real axis.

Thus,

$$\int_C \frac{f(z)}{g(z)} dz = \int_{-R}^R \frac{f(x)}{g(x)} dx + \int_S \frac{f(z)}{g(z)} dz$$

As $R \rightarrow \infty$, $\int_C \frac{f(z)}{g(z)} dz \rightarrow 0$ by Cauchy's lemma

$$= \int_C \frac{f(z)}{g(z)} dz = \int_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx$$

Now applying Cauchy's Residue theorem, we can evaluate the integral on the left side.

Note: Cauchy's lemma: If $f(z)$ is continuous function such that $|zf(z)| \rightarrow 0$ uniformly as $|z| \rightarrow \infty$ on S ,

then $\int_C f(z) dz \rightarrow 0$ as $R \rightarrow \infty$, where ' S ' is semicircle of radius ' R ' above the real axis.

Problems based on Contour Integration

Example: 4.66 Evaluate $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$ where $a > b > 0$

Solution:

Replacement put $x = z \Rightarrow dx = dz$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \int_C \frac{z^2 dz}{(z^2+a^2)(z^2+b^2)} \text{ where}$$

Where C is the upper semi circle

$$= \int_C f(z) dz \quad \dots (1)$$

$$\text{Where, } f(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}$$

To find the poles, put $(z^2 + a^2)(z^2 + b^2) = 0$

$$\Rightarrow z = \pm ai, z = \pm bi, \text{ are poles of order one.}$$

Here $z = ai, bi$ lies in upper, half of the z -plane.

Find the residue for the inside pole

(i) When $z = ai$

$$\begin{aligned} [Res f(z)]_{z=ai} &= \lim_{z \rightarrow ai} (z - ai) \frac{z^2}{(z+ai)(z-ai)(z^2+b^2)} \\ &= \frac{-a^2}{2ai(b^2-a^2)} \\ &= \frac{a}{2i(a^2-b^2)} \end{aligned}$$

(ii) When $z = bi$

$$\begin{aligned} [Res f(z)]_{z=bi} &= \lim_{z \rightarrow bi} (z - bi) f(z) \\ &= \lim_{z \rightarrow bi} (z - bi) \frac{z^2}{(z^2+a^2)(z+bi)(z-bi)} \\ &= -\frac{b^2}{(a^2-b^2)2bi} \\ &= -\frac{b}{2i(a^2-b^2)} \end{aligned}$$

$\therefore$ By Cauchy's Residue theorem

$$\begin{aligned} \int_C f(z)dz &= 2\pi i [\text{sum of residues}] \\ &= 2\pi i \left[\frac{a}{2i(a^2-b^2)} - \frac{b}{2i(a^2-b^2)} \right] \\ &= \frac{2\pi i}{2i} \left[\frac{a-b}{(a-b)(a+b)} \right] \\ &= \frac{\pi}{a+b} \end{aligned}$$

$$(1) \Rightarrow \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}$$

Example: 4.67 Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}, a > 0, b > 0$

Solution:

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

Replacement put $x = z$

$$\Rightarrow dx = dz$$

$$\therefore \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

Where C is the upper semi circle

$$= \frac{1}{2} \int_C f(z)dz \dots (1)$$

$$\text{Where, } f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$$

To find the poles, put $(z^2 + a^2)(z^2 + b^2) = 0$

$$\Rightarrow z = \pm ai, \pm bi \text{ are poles of order one.}$$

Here $z = ai, bi$ lies in the upper half of the z - plane.

Find the residue for the inside pole

(i) When $z = ai$

$$[Res f(z)]_{z=ai} = \lim_{z \rightarrow ai} (z - ai) f(z)$$

$$\begin{aligned}
&= \lim_{z \rightarrow ai} (z - ai) \frac{1}{(z+ai)(z-ai)(z^2+b^2)} \\
&= \frac{1}{2ai(b^2-a^2)} = -\frac{1}{2ai(a^2-b^2)}
\end{aligned}$$

(ii) When $z = bi$

$$\begin{aligned}
[Res f(z)]_{z=bi} &= \lim_{z \rightarrow bi} (z - bi)f(z) \\
&= \lim_{z \rightarrow bi} (z - bi) = \frac{1}{(z^2+a^2)(z+bi)(z-bi)} \\
&= \frac{1}{(a^2-b^2)2bi}
\end{aligned}$$

$\therefore$ By Cauchy's Residue theorem

$$\begin{aligned}
\int_C f(z)dz &= 2\pi i \left[-\frac{1}{2ai(a^2-b^2)} + \frac{1}{2bi(a^2-b^2)} \right] \\
&= \frac{2\pi i}{2i(a^2-b^2)} \left[-\frac{1}{a} + \frac{1}{b} \right] \\
&= \frac{\pi}{(a+b)(a-b)} \left(\frac{a-b}{ab} \right) \\
&= \frac{\pi}{ab(a+b)} \\
(1) \Rightarrow \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} &= \frac{1}{2} \frac{\pi}{ab(a+b)} \\
&= \frac{\pi}{2ab(a+b)}
\end{aligned}$$

Example: 4.68 Evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2}$

Solution:

Replacement put $x = z$

$$\Rightarrow dx = dz$$

$$\begin{aligned}
\therefore \int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} &= \int_C \frac{dz}{z^2+a^2} \text{ where } C \text{ is the upper semi circle} \\
&= \int_C f(z)dz \quad \dots (1)
\end{aligned}$$

$$\text{Where, } f(z) = \frac{1}{z^2+a^2}$$

To find the poles put $z^2 + a^2 = 0$

$$\Rightarrow z \pm ai \text{ are poles of order one.}$$

Here $z = ai$ lies in the upper half of z plane.

Find the residue for the inside pole.

(i) When $z = ai$

$$\begin{aligned}
[Res f(z)]_{z=ai} &= \lim_{z \rightarrow ai} (z - ai)f(z) \\
&= \lim_{z \rightarrow ai} (z - ai) \frac{1}{(z+ai)(z-ai)} \\
&= \frac{1}{2ai}
\end{aligned}$$

$\therefore$ By Cauchy's Residue Theorem

$$\int_C f(z)dz = 2\pi i [\text{sum of residues}]$$

$$= 2\pi i \left[\frac{1}{2ai} \right] = \frac{\pi}{a}$$

$$(1) \Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \frac{\pi}{a}$$

Example: 4.69 Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}$

Solution:

Replacement put $x = z \Rightarrow dx = dz$

$$\text{Now, } \int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}$$

$$= \frac{1}{2} \int_C \frac{dz}{(z^2+a^2)^2} \text{ where } C \text{ is the upper semi circle}$$

$$= \frac{1}{2} \int_C f(z) dz \quad \dots (1)$$

$$\text{Where, } f(z) = \frac{1}{(z^2+a^2)^2}$$

To find the poles, put $(z^2 + a^2)^2 = 0$

$\Rightarrow z = \pm ai$ are poles of order 2 here $z = ai$ lies in the upper half of z - plane. Find the residue of the inside pole.

(i) When $z = ai$

$$\begin{aligned} [Res f(z)]_{z=ai} &= \lim_{z \rightarrow ai} \frac{d}{dz} (z - ai)^2 f(z) \\ &= \lim_{z \rightarrow ai} \frac{d}{dz} \left[(z - ai)^2 \frac{1}{(z - ai)^2 (z + ai)^2} \right] \\ &= \lim_{z \rightarrow ai} \frac{d}{dz} \left[\frac{1}{(z + ai)^2} \right] \\ &= \lim_{z \rightarrow ai} \left[\frac{-2}{(z + ai)^3} \right] \\ &= -\frac{2}{(2ai)^3} = -\frac{2}{-8a^3i} = \frac{1}{4ia^3} \end{aligned}$$

$\therefore$ By Cauchy's Residue theorem,

$$\begin{aligned} \int_C f(z) dz &= 2\pi i [\text{sum of residues}] \\ &= 2\pi i \left(\frac{1}{4ia^3} \right) \\ &= \frac{\pi}{2a^3} \end{aligned}$$

Example: 4.70 Evaluate $\int_{-\infty}^{\infty} \frac{x^2-x+2}{(x^4+10x^2+9)} dx$

Solution:

Replacement Put $x = z \Rightarrow dx = dz$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2-x+2}{(x^4+10x^2+9)} dx = \int_C \frac{z^2-z+2}{(z^4+10z^2+9)} dz, \text{ where } C \text{ is the upper semi circle.}$$

$$= \int_C f(z) dz \quad \dots (1)$$

$$\text{Where, } f(z) = \frac{z^2-z+2}{(z^4+10z^2+9)}$$

To find the poles, put $z^4 + 10z^2 + 9 = 0$

$$\Rightarrow (z^2 + 1)(z^2 + 9) = 0$$

$$\Rightarrow z = \pm i, \pm 3i \text{ are poles of order one.}$$

Here $z = i, 3i$ lies in the inside pole

Find the residue of the inside pole.

(i) When $z = i$

$$\begin{aligned} [\text{Res } f(z)]_{z=i} &= \lim_{z \rightarrow i} (z - i) f(z) \\ &= \lim_{z \rightarrow i} \left[(z - i) \frac{(z^2 - z + 2)}{(z + i)(z - i)(z^2 + 9)} \right] \\ &= \lim_{z \rightarrow i} \left[\frac{(z^2 - z + 2)}{(z + i)(z^2 + 9)} \right] \\ &= \frac{-1 - i + 2}{(2i)(8)} = \frac{1 - i}{16i} \end{aligned}$$

(ii) When $z = 3i$

$$\begin{aligned} [\text{Res } f(z)]_{z=3i} &= \lim_{z \rightarrow 3i} \frac{d}{dz} (z - 3i) f(z) \\ &= \lim_{z \rightarrow 3i} \left[(z - 3i) \frac{(z^2 - z + 2)}{(z^2 + 1)(z + 3i)(z - 3i)} \right] \\ &= \lim_{z \rightarrow 3i} \frac{(z^2 - z + 2)}{(z^2 + 1)(z + 3i)} \\ &= \frac{-9 - 3i + 2}{(-8)(6i)} = \frac{-7 - 3i}{-48i} \\ &= \frac{7 + 3i}{48i} \end{aligned}$$

∴ By Cauchy's Residue theorem,

$$\begin{aligned} \int_C f(z) dz &= 2\pi i [\text{sum of residues}] \\ &= 2\pi i \left(\frac{1 - i}{16i} + \frac{7 + 3i}{48i} \right) \\ &= 2\pi i \left(\frac{3 - 3i + 7 + 3i}{48i} \right) \\ &= 2\pi i \left(\frac{10}{48i} \right) = \frac{5\pi}{12} \end{aligned}$$

$$(1) \Rightarrow \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{(x^4 + 10x^2 + 9)} dx = \frac{5\pi}{12}$$

Example: 4.71 Evaluate $\int_0^{\infty} \frac{dx}{x^4 + a^4}$

Solution:

$$\int_0^{\infty} \frac{dx}{x^4 + a^4} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4}$$

Replacement Put $x = z \Rightarrow dx = dz$

$$\begin{aligned} \therefore \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4} &= \frac{1}{2} \int_C \frac{dz}{z^4 + a^4} \text{ where } C \text{ is the upper semi circle.} \\ &= \frac{1}{2} \int_C f(z) dz \quad \dots (1) \end{aligned}$$

$$\text{Where, } f(z) = \frac{1}{z^4 + a^4}$$

To find the poles, put $z^4 + a^4 = 0$

$$\Rightarrow z^4 = -a^4$$

$$\begin{aligned}
\Rightarrow z &= (-a^4)^{\frac{1}{4}} \\
\Rightarrow z &= (-1)^{1/4} a \\
&= (\cos \pi + i \sin \pi)^{\frac{1}{4}} a \\
&= [\cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi)]^{\frac{1}{4}} a \\
&= \left[\cos\left(\frac{\pi+2k\pi}{4}\right) + i \sin\left(\frac{\pi+2k\pi}{4}\right) \right] a \\
&= a e^{+i\left(\frac{\pi+2k\pi}{4}\right)} ; k = 0, 1, 2, 3 \dots
\end{aligned}$$

When $k = 0, z = a e^{\frac{i\pi}{4}}$

When $k = 1, z = a e^{\frac{3i\pi}{4}}$

When $k = 2, z = a e^{\frac{5i\pi}{4}}$

When $k = 3, z = a e^{\frac{7i\pi}{4}}$ are all poles of order one.

Here $z = a e^{\frac{i\pi}{4}}$ and $z = a e^{\frac{3i\pi}{4}}$ lies in the upper half of the z plane.

Find the residue for the inside pole

(i) When $z = a e^{\frac{i\pi}{4}}$

$$\begin{aligned}
[\text{Res } f(z)]_{z \rightarrow a e^{\frac{i\pi}{4}}} &= \lim_{z \rightarrow a e^{\frac{i\pi}{4}}} \left(z - a e^{\frac{i\pi}{4}} \right) f(z) \\
&= \lim_{z \rightarrow a e^{\frac{i\pi}{4}}} \left[\left(z - a e^{\frac{i\pi}{4}} \right) \frac{1}{(z^4 + a^4)} \right] \\
&= \frac{0}{0} \text{ [Apply L'Hospital rule]} \\
&= \lim_{z \rightarrow a e^{\frac{i\pi}{4}}} \frac{1}{4z^3} \\
&= \frac{1}{4a^3 e^{\frac{3i\pi}{4}}}
\end{aligned}$$

(ii) When $z = a e^{\frac{3i\pi}{4}}$

$$\begin{aligned}
[\text{Res } f(z)]_{z \rightarrow a e^{\frac{3i\pi}{4}}} &= \lim_{z \rightarrow a e^{\frac{3i\pi}{4}}} \left(z - a e^{\frac{3i\pi}{4}} \right) f(z) \\
&= \lim_{z \rightarrow a e^{\frac{3i\pi}{4}}} \left(z - a e^{\frac{3i\pi}{4}} \right) \frac{1}{z^4 + a^4} \\
&= \frac{0}{0} \text{ [Apply L'Hospital rule]} \\
&= \lim_{z \rightarrow a e^{\frac{3i\pi}{4}}} \frac{1}{4z^3} \\
&= \frac{1}{4a^3 e^{\frac{9i\pi}{4}}}
\end{aligned}$$

$\therefore$ By Cauchy's Residue theorem,

$$\int_C f(z) dz = 2\pi i [\text{sum of residues}]$$

$$\begin{aligned}
&= 2\pi i \left(\frac{1}{4a^3 e^{\frac{3i\pi}{4}}} + \frac{1}{4a^3 e^{\frac{9i\pi}{4}}} \right) \\
&= \frac{2\pi i}{4a^3} \left(e^{\frac{-i3\pi}{4}} + e^{\frac{-i9\pi}{4}} \right) \\
&= \frac{\pi i}{2a^3} \left(e^{-\pi i} e^{\frac{i\pi}{4}} + e^{-2\pi i} e^{-\frac{i\pi}{4}} \right) \quad [\because e^{-\pi i} = -1] \\
&= \frac{\pi i}{2a^3} \left((-1) e^{\frac{i\pi}{4}} + (-1) e^{-\frac{i\pi}{4}} \right) \quad [\because e^{-2\pi i} = -1] \\
&= \frac{-\pi i}{a^3} \left(\frac{e^{\frac{i\pi}{4}} - e^{-\frac{i\pi}{4}}}{2} \right) \quad \left[\because \frac{e^{ix} - e^{-ix}}{2} = i \sin x \right] \\
&= \frac{-\pi i}{a^3} \left(i \sin \frac{\pi}{4} \right) \\
&= \frac{\pi}{a^3} \left(\frac{1}{\sqrt{2}} \right) \\
(1) \Rightarrow \int_0^\infty \frac{dx}{(x^4 + a^4)} &= \frac{1}{2} \left(\frac{\pi}{\sqrt{2}a^3} \right)
\end{aligned}$$

Example: 4.72 Evaluate $\int_{-\infty}^\infty \frac{dx}{(x^2+1)^3}$

Solution:

Replacement Put $x = z \Rightarrow dx = dz$

$$\therefore \int_{-\infty}^\infty \frac{dx}{(x^4+1)^3} = \int_C \frac{dz}{(z^4+1)^3} \quad \text{where } C \text{ is the upper semi circle.}$$

$$= \int_C f(z) dz \quad \dots (1)$$

$$\text{Where, } f(z) = \frac{1}{(z^4+1)^3}$$

To find the poles, put $(z^2 + 1)^3 = 0$

$$\Rightarrow z^2 + 1 = 0$$

$$\Rightarrow z = \pm i \text{ are poles of order 3.}$$

Here $z = i$ lies in the upper half of z - plane.

Find the residue for the inside pole $z = i$

$$\begin{aligned}
[Res f(z)]_{z=i} &= \frac{1}{2} \lim_{z \rightarrow i} \frac{d^2}{dz^2} (z-i)^3 f(z) \\
&= \frac{1}{2!} \lim_{z \rightarrow i} \frac{d^2}{dz^2} \left[(z-i)^3 \frac{1}{(z+i)^3 (z-i)^3} \right] \\
&= \frac{1}{2!} \lim_{z \rightarrow i} \frac{d^2}{dz^2} \left[\frac{1}{(z+i)^3} \right] \\
&= \frac{1}{2} \lim_{z \rightarrow i} \frac{d}{dz} \left[\frac{-3}{(z+i)^4} \right] \\
&= \frac{1}{2} \frac{12}{(2i)^5} = \frac{6}{32i} = \frac{3}{16i}
\end{aligned}$$

$\therefore$ By Cauchy's Residue theorem,

$$\int_C f(z) dz = 2\pi i [\text{sum of residues}]$$

$$= 2\pi i \left(\frac{3}{16i} \right) = \frac{3\pi}{8}$$

$$(1) \Rightarrow \int_{-\infty}^\infty \frac{dx}{(x^4+1)^3} = \frac{3\pi}{8}$$

Type III

Integrals of the form

$$\int_{-\infty}^{\infty} \frac{f(x)}{g(x)} \sin(nx) dx \text{ (or) } \int_{-\infty}^{\infty} \frac{f(x)}{g(x)} \cos(nx) dx$$

To evaluate this integral, write $\sin(nx)$ and $\cos(nx)$ in terms of e^{inx} thus,

$$\int_C \frac{f(z)}{g(z)} e^{inz} dz = \int_{-\infty}^{\infty} \frac{f(x)}{g(x)} e^{inx} dx$$

Where C is the closed curve as in type II and finally equate imaginary part or real part accordingly to get the required integral.

Problems based on Contour Integration

Example: 4.73 Evaluate $\int_0^{\infty} \frac{\cos mx}{x^2+a^2} dx, a > 0, m > 0$

Solution:

Replacement put $x = z \Rightarrow dx = dz$ and $\cos mn = R.P e^{imn}$

$$\begin{aligned} \text{Now, } \int_0^{\infty} \frac{\cos mx}{x^2+a^2} dx &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{RP e^{imx}}{x^2+a^2} dx \\ &= \frac{1}{2} \int_C \frac{RP e^{imz}}{z^2+a^2} dz \text{ where C is the upper semi circle.} \\ &= \frac{R.P}{2} \int_C f(z) dz \quad \dots (1) \end{aligned}$$

$$\text{Where } f(z) = \frac{e^{imz}}{z^2+a^2}$$

To find the poles, put $z^2 + a^2 = 0$

$\Rightarrow z = \pm ai$ are poles of order one.

Here $z = ai$ lies in the upper half of z - plane.

Find the residue for the inside pole $z = ai$

$$\begin{aligned} [Res f(z)]_{z=ai} &= \lim_{z \rightarrow ai} (z - ai) f(z) \\ &= \lim_{z \rightarrow ai} (z - ai) \frac{e^{imz}}{(z+ai)(z-ai)} \\ &= \lim_{z \rightarrow ai} \frac{e^{imz}}{(z+ai)} \\ &= \frac{e^{-ma}}{2ai} \end{aligned}$$

$\therefore$ By Cauchy's Residue theorem,

$$\begin{aligned} \int_C f(z) dz &= 2\pi i [\text{sum of residues}] \\ &= 2\pi i \left(\frac{e^{-ma}}{2ai} \right) \\ &= \frac{\pi e^{-ma}}{a} \end{aligned}$$

$$(1) \Rightarrow \int_0^{\infty} \frac{\cos mx}{x^2+a^2} dx = \frac{R.P}{2} \left(\pi \frac{e^{-ma}}{a} \right) = \frac{\pi}{2a} e^{-ma}$$

Example: 4.74 Evaluate $\int_0^{\infty} \frac{x \sin mx}{x^2+a^2} dx$ where $a > 0, m > 0$

Solution:

Replacement put $x = z \Rightarrow dx = dz$ and $\sin(mx) = IP e^{imx}$

$$\begin{aligned}
\text{Now, } \int_0^\infty \frac{x \sin mx}{x^2+a^2} dx &= \frac{1}{2} \int_{-\infty}^\infty \frac{xIP e^{imx}}{x^2+a^2} dx \\
&= \frac{I.P}{2} \int_C \frac{z e^{imz}}{z^2+a^2} dz \text{ where } C \text{ is the upper semi circle.} \\
&= \frac{I.P}{2} \int_C f(z) dz \quad \dots (1)
\end{aligned}$$

$$\text{Where, } f(z) = \frac{z e^{imz}}{z^2+a^2}$$

To find the poles, put $f(z)$, put $z^2 + a^2 = 0$

$$\Rightarrow z = \pm ai \text{ are poles of order one.}$$

Here $z = ai$ lies in the upper half of z - plane.

Find the residue for the inside pole $z = ai$

$$\begin{aligned}
[Res f(z)]_{z=ai} &= \lim_{z \rightarrow ai} (z - ai) f(z) \\
&= \lim_{z \rightarrow ai} (z - ai) \frac{z e^{imz}}{(z+ai)(z-ai)} \\
&= \frac{(ai)e^{-ma}}{2ai} = \frac{e^{-ma}}{2}
\end{aligned}$$

$\therefore$ By Cauchy's Residue theorem,

$$\begin{aligned}
\int_C f(z) dz &= 2\pi i [\text{sum of residues}] \\
&= 2\pi i \left(\frac{e^{-ma}}{2} \right) = \pi e^{-ma}
\end{aligned}$$

$$(1) \Rightarrow \int_0^\infty \frac{x \sin mx}{x^2+a^2} dx = \frac{I.P}{2} (\pi i e^{-ma}) = \frac{\pi}{2} e^{-ma}$$

Example: 4.75 Evaluate $\int_{-\infty}^\infty \frac{\cos x \, dx}{(x^2+a^2)(x^2+b^2)}, a > b > 0$

Solution:

$$\text{Replacement put } n = z \Rightarrow dz = dz \cos x = R.P e^{ix}$$

$$\text{Now, } \int_{-\infty}^\infty \frac{\cos x}{x^2+a^2} \frac{dx}{(x^2+b^2)} = \int_C \frac{RP e^{iz} dz}{(z^2+a^2)(z^2+b^2)}$$

where C is the upper semi circle.

$$= \frac{R.P}{2} \int_C f(z) dz$$

$$\text{Where, } f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$$

To find the poles, put $f(z)$, put $(z^2 + a^2)(z^2 + b^2) = 0$

$\Rightarrow z = \pm ai, \pm bi$ are poles of order one here $z = ai, bi$ lies in the upper half of z - plane.

To find the residue for the inside pole

(i) when $z = ai$

$$\begin{aligned}
[Res f(z)]_{z=ai} &= \lim_{z \rightarrow ai} (z - ai) f(z) \\
&= \lim_{z \rightarrow ai} (z - ai) \frac{e^{iz}}{(z+ai)(z-ai)(z^2+b^2)} \\
&= \lim_{z \rightarrow ai} \frac{e^{iz}}{(z+ai)(z^2+b^2)} \\
&= \frac{e^{-a}}{(2ai)(b^2-a^2)} = \frac{-e^{-a}}{(2ai)(a^2-b^2)}
\end{aligned}$$

(ii) when $z = bi$

$$\begin{aligned}
 [Res f(z)]_{z=bi} &= \lim_{z \rightarrow bi} (z - bi) f(z) \\
 &= \lim_{z \rightarrow bi} (z - bi) \frac{e^{iz}}{(z^2 + a^2)(z + bi)(z - bi)} \\
 &= \lim_{z \rightarrow bi} \frac{e^{iz}}{(z^2 + a^2)(z + bi)} \\
 &= \frac{e^{-b}}{2bi(a^2 - b^2)}
 \end{aligned}$$

∴ By Cauchy's Residue theorem,

$$\begin{aligned}
 \int_C f(z) dz &= 2\pi i [\text{sum of residues}] \\
 &= 2\pi i \left(\frac{e^{-a}}{(2ai)(a^2 - b^2)} + \frac{e^{-b}}{(2bi)(a^2 - b^2)} \right) \\
 &= \frac{2\pi i}{(2i)(a^2 - b^2)} \left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right] \\
 &= \frac{\pi}{(a^2 - b^2)} \left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right]
 \end{aligned}$$

$$(1) \Rightarrow \int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2 + a^2)(x^2 + b^2)} R.P. \frac{\pi}{a^2 - b^2} \left(\frac{ae^{-b} - be^{-a}}{ab} \right) = \frac{\pi}{ab(a^2 - b^2)} (ae^{-b} - be^{-a})$$

Exercise: 4.4

Evaluate the following integrals

- | | |
|--|---|
| 1) $\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)(x^2 + 4)} dx$ | Ans: $\frac{\pi}{3}$ |
| 2) $\int_0^{\infty} \frac{x^2 \, dx}{(x^2 + 9)(x^2 + 4)}$ | Ans: $\frac{\pi}{10}$ |
| 3) $\int_0^{\infty} \frac{dx}{(x^2 + 1)^2}$ | Ans: $\frac{\pi}{4}$ |
| 4) $\int_0^{\infty} \frac{\cos 3x \, dx}{(x^2 + 1)(x^2 + 4)}$ | Ans: $\frac{\pi}{6} \left[\frac{1}{e^3} - \frac{e}{2e^6} \right]$ |
| 5) $\int_0^{\infty} \frac{\cos 3x \, dx}{(x^2 + 1)(x^2 + 4)}$ | Ans: $\frac{\pi}{6} \left[\frac{1}{e^3} - \frac{e}{2e^6} \right]$ |
| 6) $\int_0^{\infty} \frac{x^2}{(x^2 + 1)} dx$ | Ans: $\frac{\pi}{4}$ |
| 7) $\int_0^{\infty} \frac{dx}{(x^2 + a^2)^3}$ | Ans: $\frac{3\pi}{16a^5}$ |
| 8) $\int_0^{\infty} \frac{dx}{x^4 + 10x^2 + 9}$ | Ans: $\frac{\pi}{24}$ |
| 9) $\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2 + 1)} dx$ | Ans: $\frac{2\pi}{e^3}$ |
| 10) $\int_0^{\infty} \frac{x \sin x}{x^2 + 1} dx$ | Ans: $\frac{\pi}{2e}$ |
| 11) $\int_{-\infty}^{\infty} \frac{\sin x}{x^4 + 4x^2 + 5} dx$ | Ans: $\frac{-\pi \sin 2}{e}$ |
| 12) $\int_0^{\infty} \frac{2x^2 - 1}{x^4 + 5x^2 + 4} dx$ | Ans: $\frac{\pi}{4}$ |