

2.4 Recurrence Relations:

An equation that expresses a_n , the general term of the sequence $\{a_n\}$ in terms of one or more of the previous terms of the sequence, namely a_0, a_1, \dots, a_{n-1} , for all integers n with $n \geq n_0$, where n_0 is a non – negative integer is called a recurrence relation for $\{a_n\}$ or a difference equation.

If the terms of a sequence satisfies a recurrence relation, then the sequence is called a solution of the recurrence relation.

For example, we consider the famous Fibonacci sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, ...

Which can be represented by the recurrence relation.

$$F_n = F_{n-1} + F_{n-2}, n \geq 2$$

and $F_0 = 0, F_1 = 1$

Here, $F_0 = 0, F_1 = 1$ are called initial conditions.

It is a second order recurrence relation.

Definition:

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_K a_{n-k}$$

Where C_1, C_2, \dots, C_k are real numbers, and $C_k \neq 0$.

The recurrence relation in the definition is linear since the right – hand side is a sum of multiplies of the previous terms of the sequence.

The recurrence relation is homogeneous, since no terms occur that are not multiplies of the a_j 's.

The coefficients of the terms of the sequence are all constants, rather than function that depend on “ n ”.

The degree is k because a_n is expressed in terms of the previous k terms of the sequence.

Solving Linear Homogeneous Recurrence Relations With Constant Coefficients:

Step: 1 Write down the characteristic equation for the given recurrence relation.

Here, the degree of character equation is 1 less than the number of terms in recurrence relation.

Step: 2 By solving the characteristic equation find out the characteristic roots.

Step: 3 Depends upon the nature of roots, find out the solution a_n as follows:

Case (i) Let the roots be real and distinct say r_1, r_2, \dots, r_n .

Then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n + \dots + \alpha_n r_n^n$ where $\alpha_1, \alpha_2, \dots, \alpha_n$ are arbitrary constants.

Case (ii) Let the roots be real and equal say $r_1 = r_2 = \dots = r_n$.

Then $a_n = \alpha_1 r_1^n + n\alpha_2 r_2^n + n^2\alpha_3 r_3^n + \dots + n^n\alpha_n r_n^n$ where $\alpha_1, \alpha_2, \dots, \alpha_n$ are arbitrary constants.

Case (iii) When the roots are complex conjugate, then

$$a_n = r^n(\alpha_1 \cos n\theta + \alpha_2 \sin n\theta)$$

Step: 4 Apply initial conditions and find out arbitrary constants.

Note:

There is no single method or technique to solve all recurrence relations. There exist some recurrence relations which cannot be solved. The recurrence relation

$$S(k) = 2[S(k-1)]^2 - kS(k-3) \text{ cannot be solved.}$$

1. If the sequence $a_n = 3 \cdot 2^n, n \geq 1$, then find the corresponding recurrence relation.

Solution:

$$\text{Given } a_n = 3 \cdot 2^n$$

$$\Rightarrow a_{n-1} = 3 \cdot 2^{n-1}$$

$$= 3 \cdot \frac{2^n}{2}$$

$$\Rightarrow a_{n-1} = \frac{a^n}{2}$$

$$\Rightarrow a_n = 2(a_{n-1})$$

Hence $a_n = 2a_{n-1}, n \geq 1$ with $a_0 = 3$

2. Find the recurrence relation for $S(n) = 6(-5)^n, n \geq 0$

Solution:

Given $S(n) = 6(-5)^n$

$$\Rightarrow S(n-1) = 6(-5)^{n-1}$$

$$= 6 \frac{(-5)^n}{-5}$$

$$= \frac{S(n)}{-5}$$

$$\Rightarrow S(n) = -5 \cdot S(n-1), n \geq 0 \text{ with } S(0) = 6$$

3. Find the recurrence relation from $y_k = A \cdot 2^k + B \cdot 3^k$

Solution:

Given $y_k = A \cdot 2^k + B \cdot 3^k \quad \dots (1)$

$$\Rightarrow y_{k+1} = A \cdot 2^{k+1} + B \cdot 3^{k+1}$$

$$= A \cdot 2^k \cdot 2 + B \cdot 3^k \cdot 3$$

$$= 2A \cdot 2^k + 3B \cdot 3^k \quad \dots (2)$$

$$\Rightarrow y_{k+2} = 4A \cdot 2^k + 9B \cdot 3^k \quad \dots (3)$$

$$(3) - 5(2) + 6(1)$$

$$\Rightarrow y_{k+2} - 5y_{k+1} + 6y_k = 4A \cdot 2^k + 9B \cdot 3^k - 10A \cdot 2^k - 15B \cdot 3^k + 6A \cdot 2^k + 6B \cdot 3^k = 0$$

$$\Rightarrow y_{k+2} - 5y_{k+1} + 6y_k = 0$$

4. Find the recurrence relation from $y_n = A3^n + B(-4)^n$

Solution:

$$\text{Given } y_n = A3^n + B(-4)^n \quad \dots (1)$$

$$\Rightarrow y_{n+1} = y_n = A3^{n+1} + B(-4)^{n+1}$$

$$= A3^n \cdot 3 + B(-4)^n \cdot (-4)$$

$$= 3A \cdot 3^n - 4B \cdot (-4)^n \quad \dots (2)$$

$$\Rightarrow y_{n+2} = 9A \cdot 3^n + 16B \cdot (-4)^n \quad \dots (3)$$

$$(3) + (2) - 12(1)$$

$$\Rightarrow y_{n+2} + y_{n+1} - 12y_n = 9A3^n + 16B(-4)^n + 3A3^n - 4B(-4)^n - 12A3^n - 12B(-4)^n = 0$$

$$\Rightarrow y_{n+2} + y_{n+1} - y_n = 0$$

5. Find the solution to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$

with the initial conditions $a_0 = 2, a_1 = 5, a_2 = 15$

Solution:

The recurrence relation can be written as $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$

The characteristic equation is $r^3 - 6r^2 + 11r - 6 = 0$

By solving, we get the characteristic roots, $r = 1, 2, 3$

Solution is $a_n = \alpha_1 \cdot 1^n + \alpha_2 2^n + \alpha_3 3^n \dots (A)$

Given $a_0 = 2$, Put $n = 0$ in (A)

$$a_0 = \alpha_1 \cdot (1)^0 + \alpha_2(2)^0 + \alpha_3(3)^0$$

$$(A) \Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = 2 \dots (1)$$

Given $a_1 = 5$, Put $n = 1$ in (A)

$$a_1 = \alpha_1 \cdot (1)^1 + \alpha_2(2)^1 + \alpha_3(3)^1$$

$$(A) \Rightarrow \alpha_1 + 2\alpha_2 + 3\alpha_3 = 5 \dots (2)$$

Given $a_2 = 15$, Put $n = 2$ in (A)

$$a_2 = \alpha_1 \cdot (1)^2 + \alpha_2(2)^2 + \alpha_3(3)^2$$

$$(A) \Rightarrow \alpha_1 + 4\alpha_2 + 9\alpha_3 = 15 \quad \dots (3)$$

To solve (1), (2) and (3)

$$(1) \Rightarrow \alpha_3 = 2 - \alpha_1 - \alpha_2 \quad \dots (4)$$

Using (4) in (2)

$$(2) \Rightarrow 2\alpha_1 + \alpha_2 = 1 \quad \dots (5)$$

Using (4) in (3)

$$(3) \Rightarrow 8\alpha_1 + 5\alpha_3 = 3 \quad \dots (6)$$

Solving (5) and (6), we get $\alpha_1 = 1$ and $\alpha_2 = -1$

Using $\alpha_1 = 1$ and $\alpha_2 = -1$ in (1) we get $\alpha_3 = 2$

Substituting $\alpha_1 = 1$ and $\alpha_2 = -1$ and $\alpha_3 = 2$ in (A), we get

Solution is $a_n = 1 \cdot 1^n - 1 \cdot 2^n + 2 \cdot 3^n$