ANALYTIC FUNCTIONS – NECESSARY AND SUFFICIENT CONDITIONS FOR ANALYTICITY IN CARTESIAN AND POLAR CO-ORDINATES

Analytic [or] Holomorphic [or] Regular function

A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

Entire Function: [Integral function]

A function which is analytic everywhere in the finite plane is called an entire function.

An entire function is analytic everywhere except at $z = \infty$.

Example: $e^z, \sin z, \cos z, sinhz, \cosh z$

Example: Show that $f(z) = log\ z$ analytic everywhere except at the origin and find its derivatives.

Solution:

$$\text{Let } z = re^{i\theta}$$

$$f(z) = log\ z$$

$$= \log(re^{i\theta}) = logr + \log(e^{i\theta}) = logr + i\theta$$

But, at the origin, $r = 0$. Thus, at the origin,

$$f(z) = log0 + i\theta = -\infty + i\theta$$

So, $f(z)$is not defined at the origin and hence is not differentiable there.

Note : $e^{-\infty} = 0$

$\log e^{-\infty} = \log 0\,; -\infty = log0$

At points other than the origin, we have

$u(r,\theta) = log\ r$	$v(r,\theta) = \theta$
$u_r = \frac{1}{r}$ $u_\theta = 0$	$v_r = 0$ $v_\theta = 1$

So, $logz$ satisfies the C−R equations.

Further $\frac{1}{r}$ is not continuous at $z = 0$.

So, $u_r, u_\theta, v_r, v_\theta$ are continuous everywhere except at $z = 0$. Thus log z satisfies all the sufficient conditions for the existence of the derivative except at the origin. The derivative is

$$f'(z) = \frac{u_r + iv_r}{e^{i\theta}} = \frac{\left(\frac{1}{r}\right) + i(0)}{e^{i\theta}} = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

Note: $f(z) = u + iv \Rightarrow f(re^{i\theta}) = u + iv$

Differentiate w.r.to 'r', we get

$$(i.e.)\ e^{i\theta} f'(re^{i\theta}) = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

Example: Check whether $w = \bar{z}$ is analytics everywhere.

Solution:

Let $w = f(z) = \bar{z}$

u+$iv = x - iy$

$u = x$	$v = -y$
$u_x = 1$	$v_x = 0$
$u_y = 0$	$v_y = -1$

$u_x \neq v_y$ at any point p(x,y)

Hence, C−R equations are not satisfied.

∴The function $f(z)$ is nowhere analytic.

Example: Test the analyticity of the function $w = \sin z$.

Solution:

Let $w = f(z) = sinz$

$$u + iv = \sin(x + iy)$$

$$u + iv = \sin x \cos iy + \cos x \sin iy$$

$$u + iv = \sin x \cosh y + i \cos x \sin hy$$

Equating real and imaginary parts, we get

$u = \sin x \cosh y$	$v = \cos x \sinh y$
$u_x = \cos x \cosh y$	$v_x = -\sin x \sinh y$
$u_y = \sin x \sinh y$	$v_y = \cos x \cosh y$

$\therefore u_x = v_y$ and $u_y = -v_x$

C −R equations are satisfied.

Also the four partial derivatives are continuous.

Hence, the function is analytic.

Example: Determine whether the function $2xy + i(x^2 - y^2)$ is analytic or not.

Solution:

Let $f(z) = 2xy + i(x^2 - y^2)$

$(i.e.)$

$u = 2xy$	$v = x^2 - y^2$
$\frac{\partial u}{\partial x} = 2y$ $\frac{\partial u}{\partial y} = 2x$	$\frac{\partial v}{\partial x} = 2x$ $\frac{\partial v}{\partial y} = -2y$

$u_x \neq v_y$ and $u_y \neq -v_x$

C−R equations are not satisfied.

Hence, $f(z)$ is not an analytic function.

Example: Prove that $f(z) = \cosh z$ is an analytic function and find its derivative.

Solution:

Given $f(z) = \cosh z = \cos(iz) = \cos[i(x + iy)]$

$= \cos(ix - y) = \cos ix \cos y + \sin(ix) \sin y$

$u + iv = \cosh x \cos y + i \sinh x \sin y$

$u = \cosh x \cos y$	$v = \sinh x \sin y$
u_x =sinh x cos y $u_y = -\cosh x \sin y$	v_x =cosh x sin y v_y =sinh x cos y

$\therefore u_x, u_y, v_x$ and v_y exist and are continuous.

$u_x = v_y$ and $u_y = -v_x$

C−R equations are satisfied.

$\therefore f(z)$ is analytic everywhere.

Now, $f'(z) = u_x + iv_x$

$= \sinh x \cos y + i \cosh x \sin y$

$= \sinh(x + iy) = \sinh z$

Example: If $w = f(z)$ is analytic, prove that $\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$ where $z = x + iy$, and prove that $\frac{\partial^2 w}{\partial z \partial \bar{z}} = 0$.

Solution:

Let $w = u(x,y) + iv(x,y)$

As $f(z)$ is analytic, we have $u_x = v_y, u_y = -v_x$

Now, $\frac{\boldsymbol{dw}}{\boldsymbol{dz}} = f'(z) = u_x + iv_x = v_y - iu_y = i(u_y + iv_y)$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}(u + iv) = -i\frac{\partial}{\partial y}(u + iv)$$

$$= \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$$

We know that, $\frac{\partial w}{\partial z} = 0$

$$\therefore \frac{\partial^2 w}{\partial z \partial \overline{z}} = 0$$

Also $\frac{\partial^2 w}{\partial \overline{z}\, \partial z} = 0$

Example: Prove that every analytic function $\boldsymbol{w = u(x,y) + iv(x,y)}$can be expressed as a function of z alone.

Proof:

Let $z = x + iy$ and $\overline{z} = x - iy$

$$x = \frac{z+\overline{z}}{2} \quad \text{and} \quad y = \frac{z+\overline{z}}{2i}$$

Hence, u and v and also w may be considered as a function of z and $\overline{z}$

Consider $\frac{\partial w}{\partial \overline{z}} = \frac{\partial u}{\partial \overline{z}} + i\frac{\partial v}{\partial \overline{z}}$

$$= \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \overline{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \overline{z}}\right) + \left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \overline{z}}\right)$$

$$= \left(\frac{1}{2}u_x - \frac{1}{2i}u_y\right) + i\left(\frac{1}{2}v_x - \frac{1}{2i}v_y\right)$$

$$= \frac{1}{2}(u_x - v_y) + \frac{i}{2}(u_y + v_x)$$

$= 0$ by C−R equations as w is analytic.

This means that w is independent of $\overline{z}$

$(i.e.)$ w is a function of z alone.

This means that if $w = u(x,y) + iv(x,y)$ is analytic, it can be rewritten as a function of $(x + iy)$.

Equivalently a function of $\overline{z}$ cannot be an analytic function of z.

Example: Find the constants a, b, c if $\mathbf{f(z)} = \boldsymbol{(x + ay) + \mathrm{i}(bx + cy)}$ is analytic.

Solution:

$f(z) = u(x,y) + iv(x,y)$

$= (x + ay) + i(bx + cy)$

$u = x + ay$	$v = bx + cy$
$u_x = 1$ $u_y = a$	$v_x = b$ $v_y = c$

Given $f(z)$ is analytic

$\Rightarrow u_x = v_y$ and $u_y = -v_x$

$1 = c$ and $a = -b$

Example: Examine whether the following function is analytic or not $f(z) = e^{-x}$(cos y – i sin y).

Solution:

Given $f(z) = e^{-x}(\cos y - i \sin y)$

$\Rightarrow u + iv = e^{-x} \cos y - ie^{-x} \sin y$

$u = e^{-x} \cos y$	$v = -e^{-x} \sin y$
$u_x = -e^{-x} \cos y$ $u_y = -e^{-x} \sin y$	$v_x = e^{-x} \sin y$ $v_y = -e^{-x} \cos y$

Here, $u_x = v_y$ and $u_y = -v_x$

$\Rightarrow$ C−R equations are satisfied

$\Rightarrow f(z)$ is analytic.

Example: Test whether the function $f(z) = \frac{1}{2}\log(x^2 + y^2 + \tan^{-1}\left(\frac{y}{x}\right)$ is analytic or not.

Solution:

Given $f(z) = \frac{1}{2}\log(x^2 + y^2 + i \tan^{-1}\left(\frac{y}{x}\right)$

$(i.e.) u + iv = \frac{1}{2}\log(x^2 + y^2 + i \tan^{-1}\left(\frac{y}{x}\right)$

$u = \frac{1}{2}\log(x^2 + y^2)$	$v = \tan^{-1}\left(\frac{y}{x}\right)$

$u_x = \frac{1}{2}\frac{1}{x^2+y^2}(2x)$ $= \frac{x}{x^2+y^2}$ $u_y = \frac{1}{2}\frac{1}{x^2+y^2}(2y)$ $= \frac{y}{x^2+y^2}$	$v_x = \frac{1}{1+\frac{y^2}{x^2}}\left[-\frac{y}{x^2}\right]$ $= \frac{-y}{x^2+y^2}$ $v_y = \frac{1}{1+\frac{y^2}{x^2}}\left[\frac{1}{x}\right]$ $= \frac{x}{x^2+y^2}$

Here, $u_x = v_y$ and $u_y = -v_x$

⇒ C−R equations are satisfied

⇒ $f(z)$ is analytic.

Example: Find where each of the following functions ceases to be analytic.

(i) $\frac{z}{(z^2-1)}$ **(ii)** $\frac{z+i}{(z-i)^2}$

Solution:

(i) Let $f(z) = \frac{z}{(z^2-1)}$

$$f'(z) = \frac{(z^2-1)(1)-z(2z)}{(z^2-1)^2} = \frac{-(z^2+1)}{(z^2-1)^2}$$

$f(z)$ is not analytic, where $f'(z)$ does not exist.

$(i.e.)\ f'(z) \to \infty$

$(i.e.)(z^2-1)^2 = 0$

$(i.e.)\quad z^2 - 1 = 0$

$z = 1$

$z = \pm 1$

∴ $f(z)$ is not analytic at the points $z = \pm 1$

(ii) Let $f(z) = \frac{z+i}{(z-i)^2}$

$$f'(z) = \frac{(z-i)^2(1)(z+i)[2(z-i)]}{(z-i)^4} = \frac{(z+3i)}{(z-i)^3}$$

$f'(z) \to \infty, at\ z = i$

∴ $f(z)$ is not analytic at $z = i$.

PROPERTIES – HARMONIC CONJUGATES

Laplace equation

$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0$ is known as Laplace equation in two dimensions.

Properties of Analytic Functions

Property: 1 Prove that the real and imaginary parts of an analytic function are harmonic functions.

Proof:

Let $f(z)=u+iv$ be an analytic function

$u_x=v_y \ldots(1)$ and $u_y=-v_x \quad \ldots(2)$ by C−R

Differentiate (1) & (2) p.w.r. to x, we get

$u_{xx}=v_{xy} \ldots(3)$ and $u_{xy}=-v_{xx} \quad \ldots(4)$

Differentiate (1) & (2) p.w.r. to x, we get

$u_{yx}=v_{yy} \ldots(5)$ and $u_{yy}=-v_{yx} \quad \ldots(6)$

(3) +(6) ⇒ $u_{xx}+u_{yy}=0 \quad [\because v_{xy}=v_{yx}]$

(5) – (4) ⇒ $v_{xx}+v_{yy}=0 \quad [\because u_{xy}=u_{yx}]$

$\therefore u$ and v satisfy the Laplace equation.

Harmonic function (or) [Potential function]

A real function of two real variables x and y that possesses continuous second order partial derivatives and that satisfies Laplace equation is called a harmonic function.

Note: A harmonic function is also known as a potential function.

Conjugate harmonic function

If u and v are harmonic functions such that $u+iv$ is analytic, then each is called the conjugate harmonic function of the other.

Property: 2 If w $=u(x,y)+iv(x,y)$ is an analytic function the curves of the family $u(x,y)=c_1$ and the curves of the family $v(x,y)=c_2$ cut orthogonally, where c_1 andc_2 are varying constants.

Proof:

Let $f(z)=u+iv$ be an analytic function

$\Rightarrow u_x=v_y \ldots(1)$ and $u_y=-v_x \quad \ldots(2)$ by C−R

Given $u=c_1$ and $v=c_2$

Differentiate p.w.r. to x, we get

$$u_x + u_y \frac{dy}{dx} = 0 \text{ and } \quad v_x + v_y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-u_x}{u_y} \text{ and } \frac{dy}{dx} = \frac{-v_x}{v_y}$$

$$\Rightarrow m_1 = \frac{-u_x}{u_y} \qquad \Rightarrow m_2 = \frac{-v_x}{v_y}$$

$$m_1.m_2 = \left(\frac{-u_x}{u_y}\right)\left(\frac{-v_x}{v_y}\right) = \left(\frac{u_x}{u_y}\right)\left(\frac{u_y}{u_x}\right) = -1 \text{ by (1) and (2)}$$

Hence, the family of curves form an orthogonal system.

Property: 3 An analytic function with constant modulus is constant.

Proof:

Let $f(z) = u + iv$ be an analytic function.

$\Rightarrow u_x = v_y \ldots(1)$ and $u_y = -v_x \quad \ldots(2)$ by C−R

Given $|f(z)| = \sqrt{u^2 + v^2} = c \neq 0$

$\Rightarrow |f(z)| = u^2 + v^2 = c^2$ (say)

$(i.e)\ u^2 + v^2 = c^2 \quad \ldots(3)$

Differentiate (3) p.w.r. to *x* and *y*; we get

$2uu_x + 2vv_x = 0 \Rightarrow uu_x + vv_x = 0 \qquad \ldots(4)$

$2uu_y + 2vv_y = 0 \Rightarrow uu_y + vv_y = 0 \qquad \ldots(5)$

(4) × u $\quad \Rightarrow u^2u_x + uv\,v_x = 0 \qquad \ldots(6)$

(5) × v $\quad \Rightarrow uv\,u_y + v^2v_y = 0 \qquad \ldots(7)$

(6)+(7) $\quad \Rightarrow u^2u_x + v^2v_y + uv\,[v_x + u_y] = 0$

$\Rightarrow u^2u_x + v^2\,u_x + uv\,[-u_y + u_y] = 0$ by (1) & (2)

$\Rightarrow (u^2{+}v^2)\,u_x = 0$

$\Rightarrow u_x = 0$

Similarly, we get $v_x = 0$

We know that $f'(z) = u_x + v_x = 0 + i0 = 0$

Integrating w.r.to z, we get, $f(z) = c$ [Constant]

Property: 4 An analytic function whose real part is constant must itself be a constant.

Proof :

Let $f(z) = u + iv$ be an analytic function.

$\Rightarrow u_x = v_y \ldots(1)$ and $u_y = -v_x \quad \ldots(2)$ by C−R

Given $u = c$ [Constant]

$\Rightarrow u_x = 0, \qquad u_y = 0$

$\Rightarrow u_x = 0, \qquad v_x = 0$ by (2)

We know that $f'(z) = u_x + iv_x = 0 + i0 = 0$

Integrating w.r.to z, we get $f(z) = c$ [Constant]

Property: 5 Prove that an analytic function with constant imaginary part is constant.

Proof:

Let $f(z) = u + iv$ be an analytic function.

$\Rightarrow u_x = v_y \ldots(1)$ and $u_y = -v_x \quad \ldots(2)$ by C−R

Given $v = c$ [Constant]

$\Rightarrow v_x = 0, \quad v_y = 0$

We know that $f'(z) = u_x + iv_x$

$= v_y + iv_x$ by (1) $= 0 + i0$

$\Rightarrow f'(z) = 0$

Integrating w.r.to z, we get $f(z) = c$ [Constant]

Property: 6 If $\mathbf{f(z)}$ and $\overline{\mathbf{f(z)}}$ are analytic in a region D, then show that $\mathbf{f(z)}$ is constant in that region D.

Proof:

Let $f(z) = u(x,y) + iv\,(x,y)$ be an analytic function.

$\overline{f(z)} = u(x,y) - iv(x,y) = u(x,y) + i[-v(x,y)]$

Since, $f(z)$ is analytic in D, we get $u_x = v_y$ and $u_y = -v_x$

Since, $\overline{f(z)}$ is analytic in D, we have $u_x = -v_y$ and $u_y = v_x$

Adding, we get $u_x = 0$ and $u_y = 0$ and hence, $v_x = v_y = 0$

$\therefore\ f(z) = u_x + iv_x = 0 + i0 = 0$

$\therefore\ f(z)$ is constant in D.

Theorem: 1 If $\mathbf{f(z)} = \boldsymbol{u + iv}$ is a regular function of z in a domain D, then $\boldsymbol{\nabla^2|f(z)|^2 = 4|f'(z)|^2}$

Solution:

Given $f(z) = u + iv$

$\Rightarrow |f(z)| = \sqrt{u^2 + v^2}$

$\Rightarrow |f(z)|^2 = u^2 + v^2$

$\Rightarrow \nabla^2|f(z)|^2 = \nabla^2(u^2 + v^2)$

$= \nabla^2(u^2) + \nabla^2(v^2) \qquad \ldots(1)$

$$\nabla^2(u^2) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2 += \frac{\partial^2(u^2)}{\partial x^2} + \frac{\partial^2(u^2)}{\partial y^2} \qquad \ldots(2)$$

$$\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}\left[2u\frac{\partial u}{\partial x}\right] = 2\left[u\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right] = 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial x}\right)^2$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^2) = 2u\frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial u}{\partial y}\right)^2$

$$(2)\Rightarrow \quad \nabla^2(u^2) = 2u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 0 + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \qquad [\because u \text{ is harmonic}]$$

$$\nabla^2(u^2) = 2u_x^2 + 2u_y^2$$

Similarly, $\nabla^2(v^2) = 2v_x^2 + 2v_y^2$

$$(1)\Rightarrow \quad \nabla^2|f(z)|^2 = 2\left[u_x^2 + u_y^2 + v_x^2 + v_y^2\right]$$

$$= 2[u_x^2 + (-v_x)^2 + v_x^2 + u_x^2] \qquad [\because u_x = v_y; u_y = -v_x]$$

$$= 4[u_x^2 + v_x^2]$$

$$(i.e.)\nabla^2|f(z)|^2 = 4|f'(z)|^2$$

Note : $f(z) = u + iv; f'(z) = u_x + iv_x$;

(or) $f'(z) = v_y + iu_y$; $|f'(z)| = \sqrt{u_x^2 + v_x^2}$; $|f'(z)|^2 = u_x^2 + v_x^2$

Theorem: 2 If $\mathbf{f(z) = u + iv}$ is a regular function of z in a domain D, then ∇^2 log $\mathbf{|f(z)| = 0}$ if $\mathbf{f(z)\, f'(z) \neq 0}$ in D. i.e., log $\mathbf{|f(z)|}$ is harmonic in D.

Solution:

Given $f(z) = u + iv$

$$|f(z)| = \sqrt{u^2 + v^2}$$

$$\log|f(z)| = \frac{1}{2}\log(u^2 + v^2)$$

$$\nabla^2\log|f(z)| = \frac{1}{2}\nabla^2\log(u^2 + v^2) = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log(u^2 + v^2)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x^2}[log(u^2 + v^2)] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[log(u^2 + v^2)] \qquad \ldots(1)$$

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}[log(u^2 + v^2)] = \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{1}{u^2+v^2}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right)\right] = \frac{\partial}{\partial x}\left[\frac{uu_x + vv_x}{u^2+v^2}\right]$$

$$= \frac{(u^2+v^2)[uu_{xx} + u_xu_x + vv_{xx} + v_xv_x] - (uu_x + vv_x)(2uu_x + 2vv_x)}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)[uu_{xx} + vv_{xx} + u_x^2 + v_x^2] - 2(uu_x + vv_x)^2}{(u^2+v^2)^2}$$

Similarly, $\frac{1}{2}\frac{\partial^2}{\partial y^2}[log(u^2 + v^2)] = \frac{(u^2+v^2)[uu_{yy} + vv_{yy} + u_y^2 + v_y^2] - 2(uu_y + vv_y)^2}{(u^2+v^2)^2}$

$$(1)\Rightarrow \quad \nabla^2\log|f(z)| = \frac{(u^2+v^2)[u(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy}) + (u_x^2 + u_y^2) + (v_x^2 + v_y^2)] - 2[uu_x + vv_x]^2 - 2[uu_y + vv_y]^2}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)[u(0)+(u_x^2+v_x^2)+u_y^2+v_y^2)--2[u^2u_x^2+v^2v_x^2+2uv\,u_xv_x+u^2\,u_y^2+v^2v_y^2+2uv\,u_y\,v_y]}{(u^2+v^2)^2}$$

$[\because u_{xx}+u_{yy}=0, v_{xx}+vv_{yy}=0]$

$$= \frac{(u^2+v^2)[|f'(z)^2|+|f'(z)^2|-2[u^2(u_x^2+u_y^2)+v^2(v_x^2+v_y^2)+2uv(u_x\,v_x+u_y\,v_y]}{(u^2+v^2)^2}$$

$[\because f'(z)=u+iv, |f'(z)|=u_x+iv_x$ (or) $f'(z)=v_y-iu_y, |f'(z)|^2=u_x^2+v_x^2$

(or) $|f'(z)|^2=u_y^2+v_y^2$

$$= \frac{2(u^2+v^2)[|f'(z)^2|-2[u^2|f'(z)|^2+v^2|f'(z)^2|+2uv(0)]}{(u^2+v^2)^2}$$

$[\because u_x = v_y,\ u_y=-v_x]$

$\Rightarrow u_x\,v_x+u_yv_y = 0$

$\Rightarrow u_x^2+u_y^2 = u_x^2+v_x^2 = |f'(z)|^2$

$\Rightarrow v_x^2+v_y^2 = u_y^2+v_y^2 = |f'(z)|^2$

$$= \frac{2(u^2+v^2)|f(z)|^2-2(u^2+v^2)|f'(z)|^2}{(u^2+v^2)^2}$$

$(i.e.)\ \nabla^2\log|f(z)| = 0$

Theorem: 3 If $f(z)=u+iv$ is a regular function of z in a domain D, then $\nabla^2(u^p)=p(p-1)\,u^{p-2}|f'(z)|^2$

Solution:

$$\nabla^2(u^p) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^p)$$

$$= \frac{\partial^2}{\partial x^2}(u^p)+\frac{\partial^2}{\partial y^2}(u^p)$$

$$\frac{\partial^2}{\partial x^2}(u^p) = \frac{\partial}{\partial x}\left[pu^{p-1}\frac{\partial u}{\partial x}\right] = pu^{p-1}u_{xx}+p(p-1)u^{p-2}(u_x)^2$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^p) = pu^{p-1}u_{yy}+p(p-1)u^{p-2}(u_y)^2$

$(1) \Rightarrow\ \nabla^2(u^p) = pu^{p-1}(u_{xx}+u_{yy})+p(p-1)u^{p-2}[u_x^2+u_y^2]$

$= pu^{p-1}(0)+p(p-1)u^{p-2}|f'(z)|^2$

$[\because u_{xx}+u_{yy}=0, f(z)=u+iv, f'(z)=u_x+iv_x,\ |f'(z)|^2=u_x^2+u_y^2$

$\therefore\ \nabla^2(u^p) = p(p-1)\,u^{p-2}|f'(z)|^2$

Theorem: 4 If $f(z)=u+iv$ is a regular function of z, then $\nabla^2|f(z)|^p = p^2|f(z)|^{p-2}|f'(z)|^2$.

Solution:

Let $f(z) = u + iv$

$$|f(z)| = \sqrt{u^2 + v^2} \qquad \ldots(a)$$

$$|f(z)|^p = (u^2 + v^2)^{p/2} \qquad \ldots(b)$$

$$\nabla^2 |f(z)|^p = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2 + v^2)^{p/2}$$

$$= \frac{\partial^2}{\partial x^2}(u^2 + v^2)^{p/2} + \frac{\partial^2}{\partial y^2}(u^2 + v^2)^{p/2}$$

$$\frac{\partial^2}{\partial x^2}(u^2 + v^2)^{p/2} = \frac{\partial}{\partial x}\left[\frac{p}{2}(u^2 + v^2)^{\frac{p}{2}-1}\left[2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right]\right]$$

$$= p(u^2 + v^2)^{\frac{p}{2}-1}[uu_{xx} + u_x u_x + vv_{xx} + v_x v_x]$$

$$+p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}(uu_x + vv_x)(2uu_x + 2vv_x)$$

$$= p(u^2 + v^2)^{\frac{p}{2}-1}[uu_{xx} + u_x^2 + vv_{xx} + v_x^2]$$

$$+2p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}(uu_x + vv_x)^2$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^2 + v^2)^{p/2} = p(u^2 + v^2)^{\frac{p}{2}-1}\left[uu_{yy} + u_y^2 + vv_{yy} + v_y^2\right]$

$$+2p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}(uu_y + vv_y)^2$$

$$\Rightarrow \nabla^2 |f(z)|^p = p(u^2 + v^2)^{\frac{p}{2}-1}\left[u(u_{xx} + u_{yy}) + v(v_{vx} + v_{yy}) + u_x^2 + u_y^2 + v_x^2 + v_y^2\right] +$$

$$2p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}\left[u^2 u_x^2 + v^2 v_x^2 + 2uv\, u_x v_x + u^2 u_y^2 + v^2 v_y^2 + 2uv\, u_y v_y\right]$$

$$= p(u^2 + v^2)^{\frac{p}{2}-1}\left[u(0) + v(0) + 2(u_x^2 + u_y^2)\right]$$

$$+2p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}\left[u^2(u_x^2 + u_y^2) + v^2(v_x^2 + v_y^2) + 2uv(u_x v_x + u_y v_y)\right]$$

$$= 2p(u^2 + v^2)^{\frac{p}{2}-1}|f'(z)|^2 + 2p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}[u^2|f'(z)|^2 + v^2|f'(z)|^2 + 2uv(0)]$$

$$= 2p(u^2 + v^2)^{\frac{p}{2}-1}|f'(z)|^2 + 2p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}(u^2 + v^2)|f'(z)|^2$$

$$= 2p(u^2 + v^2)^{\frac{p}{2}-1}|f'(z)|^2 + 2p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-1}|f'(z)|^2$$

$$= 2p(u^2 + v^2)^{\frac{p}{2}-1}|f'(z)|^2\left[1 + \frac{p}{2} - 1\right]$$

$$= 2p(u^2 + v^2)^{\frac{p}{2}-1}|f'(z)|^2 \quad = p^2(u^2 + v^2)^{\frac{p-2}{2}}|f'(z)|^2$$

$$= p^2(\sqrt{u^2 + v^2})^{p-2}|f'(z)|^2$$

$$= p^2|f(z)|^{p-2}|f'(z)|^2 \text{ by (a) \& (b)}$$

Theorem: 5 If $\mathbf{f(z) = u + iv}$ is a regular function of z, in a domain D, then

$$\left[\frac{\partial}{\partial x}|\boldsymbol{f(z)}|\right]^{\mathbf{2}} + \left[\frac{\partial}{\partial y}|\boldsymbol{f(z)}|\right]^{\mathbf{2}} = |\boldsymbol{f'(z)}|^{\mathbf{2}}$$

Solution:

Given $f(z) = u + iv$

$$|f(z)| = \sqrt{u^2 + v^2}$$

$$\frac{\partial}{\partial x}|f(z)| = \frac{\partial}{\partial x}\left[\sqrt{u^2 + v^2}\right]$$

$$= \frac{1}{2\sqrt{u^2+v^2}}[2uu_x + 2vv_x] = \frac{uu_x+vv_x}{\sqrt{u^2+v^2}}$$

$$\left[\frac{\partial}{\partial x}|f(z)|\right]^2 = \frac{(uu_x+vv_x)^2}{u^2+v^2} = \frac{u^2u_x^2+v^2v_x^2+2uv\,u_xv_x}{u^2+v^2}$$

Similarly, $\left[\frac{\partial}{\partial y}|f(z)|\right]^2 = \frac{u^2u_y^2+v^2v_y^2+2uv\,u_yv_y}{u^2+v^2}$

$$\left[\frac{\partial}{\partial x}|f(z)|\right]^2 + \left[\frac{\partial}{\partial y}|f(z)|\right]^2 = \frac{u^2[u_x^2+u_y^2]+v^2[v_x^2+v_y^2]+2uv\,[u_xv_x+u_yv_y]}{u^2+v^2}$$

$$= \frac{u^2|f'(z)|^2+v^2|f'(z)|^2+2u\,v\,(0)}{u^2+v^2}[\because u_x = v_y;\ u_y = -v_x]$$

$$= \frac{(u^2+v^2)|f(z)|^2}{u^2+v^2} = |f'(z)|^2[\because u_xv_x + u_yv_y = 0]$$

Theorem: 6 If $\mathbf{f(z)} = \boldsymbol{u + iv}$ is a regular function of z, then $\mathbf{\nabla^2|Re\ f(z)|^2 = 2|f'(z)|^2}$

Solution:

Let $f(z) = u + iv$

$$\text{Re}f(z) = u$$

$$|\text{Re}\, f'(z)|^2 = u^2$$

$$\nabla^2|\text{Re}\, f'(z)|^2 = \nabla^2u^2$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2)$$

$$= \left(\frac{\partial^2}{\partial x^2}\right)(u^2) + \left(\frac{\partial^2}{\partial y^2}\right)(u^2)$$

$$= 2[u_x^2 + u_y^2]$$

$$= 2\,|f'(z)|^2$$

Theorem: 7 If $\mathbf{f(z)} = \boldsymbol{u + iv}$ is a regular function of z, then prove that $\mathbf{\nabla^2|Im\ f(z)|^2} = \mathbf{2|f'(z)|^2}$

Proof:

Let $f(z) = u + iv$

$$\text{I m}f(z) = v$$

$$|Im\, f(z)|^2 = v^2$$

$$\frac{\partial}{\partial x}(v^2) = 2vv_x$$

$$\frac{\partial^2}{\partial x^2}(v^2) = 2[vv_{xx} + v_x v_x] = 2[vv_{xx} + v_x^2]$$

Similarly, $\frac{\partial^2}{\partial y^2}(v^2) = 2[vv_{yy} + v_y^2]$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Im\, f(z)|^2 = 2[v(v_{xx} + v_{yy}) + v_x^2 + v_y^2]$$

$$= 2[v(0) + u_x^2 + v_x^2] \qquad \text{by C−R equation}$$

$$= 2|f'(z)|^2$$

Theorem: 8 Show that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$ (or) S T $\nabla^2 = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

Proof:

Let x & y are functions of z and $\bar{z}$

that is $x = \frac{z+\bar{z}}{2}, y = \frac{z-\bar{z}}{2i}$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial}{\partial y}\left[\frac{1}{2i}\right] = \frac{1}{2}\left[\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right]$$

$$2\frac{\partial}{\partial z} = \frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y} \qquad \ldots(1)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial}{\partial y}\left[\frac{-1}{2i}\right] = \frac{1}{2}\left[\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right]$$

$$2\frac{\partial}{\partial \bar{z}} = \left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) \qquad \ldots(2)$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) \; [\because (a+b)(a-b) = a^2 - b^2]$$

$$= \left(2\frac{\partial}{\partial z}\right)\left(2\frac{\partial}{\partial \bar{z}}\right) \text{ by (1) \& (2)}$$

$$= 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

Theorem: 9 If $f(z)$ is analytic, show that $\nabla^2|f(z)|^2 = 4|f'(z)|^2$

Solution:

We know that, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

$$|f(z)|^2 = f(z)\overline{f(z)}$$

$$\nabla^2|f(z)|^2 = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}[f(z)\overline{f(z)}]$$

$$= 4\left[\frac{\partial}{\partial z}f(z)\right]\left[\frac{\partial}{\partial \bar{z}}\overline{f(z)}\right]$$

[∵ $f(z)$ is independent of $\bar{z}$ and $\overline{f(z)}$ is independent of z]

$$\therefore\ \nabla^2|f(z)|^2 = 4[f'(z)\left[\overline{\frac{\partial}{\partial z} f(z)}\right] = 4f'(z)\overline{f'(z)}$$

$$= 4|f'(z)|^2 \quad [\because z\bar{z} = |(z)|^2]$$

Example: Find the value of m if $u = 2x^2 - my^2 + 3x$ is harmonic.

Solution:

Given $u = 2x^2 - my^2 + 3x$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ [∵ u is harmonic]} \qquad \ldots(1)$$

$$\frac{\partial u}{\partial x} = 4x + 3 \qquad \frac{\partial u}{\partial y} = -2my$$

$$\frac{\partial^2 u}{\partial x^2} = 4 \qquad \frac{\partial^2 u}{\partial y^2} = -2m$$

$$\therefore (1) \Rightarrow (4) + (-2m) = 0$$

$$\Rightarrow m = 2$$