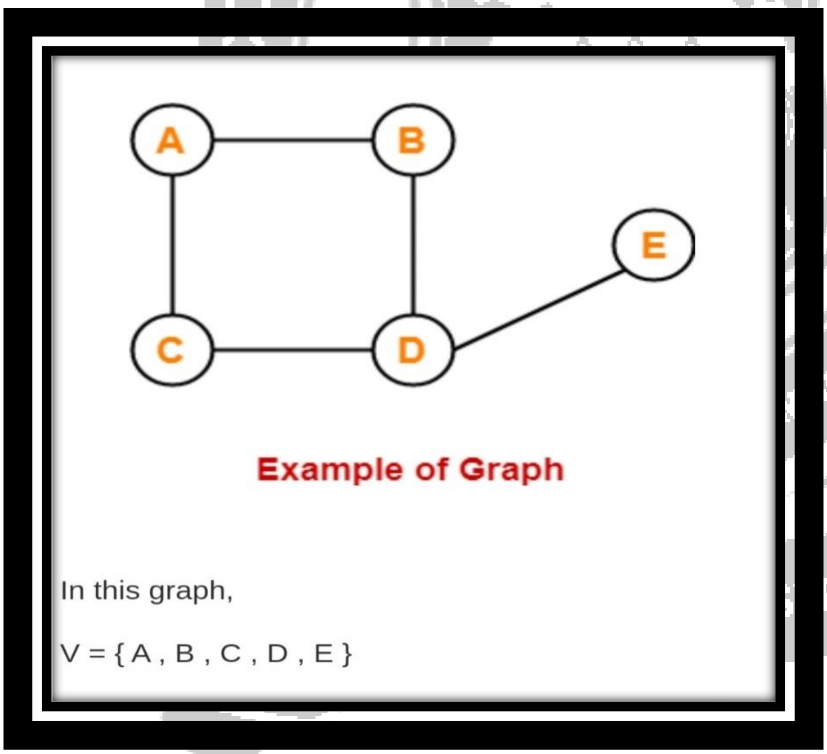
Graph:

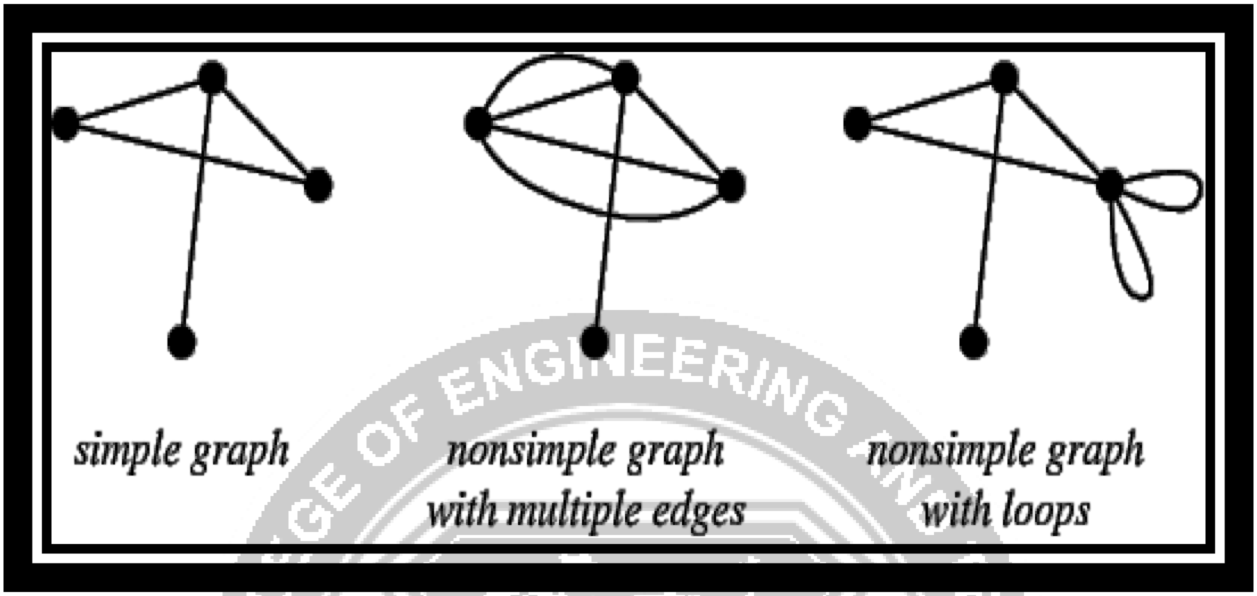
A graph $G = (V, E, \phi)$ consists of a non – empty set $V = \{V_1, V_2, \ldots\}$ called the set of nodes (Points, Vertices) of the graph, $E = \{e_1, e_2, \ldots\}$ is said to be the set of edges of the graph, and , ϕ is a mapping from the set of edges E to set of ordered or unordered pairs of elements of V.

The vertices are represented by points and each edge is represented by a line digrammatically.



Self Loop:

If there is an edge from v_i to v_i then that edge is called self loop or simply loop.



Parallel edges:

If two edges have same end points then the edges are called parallel edges.

Incident:

If the vertex v_i is an end vertex of some edge e_k then e_k is said to be incident with v_i.

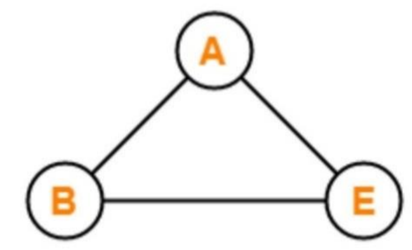
Adjacent edges and vertices:

Two edges are said to be adjacent if they are incident on a common vertex.

Two vertices v_i and v_j are said to be adjacent if $v_i v_j$ is an edge of the graph.

Simple Graph:

A graph which has neither self loops nor parallel edges is called a simple graph.



Example of Simple Graph

Isolated vertex:

A vertex having no edge incident on it is called an isolated vertex. It is obvious that for an isolated vertex degree is zero.

Pendent vertex:

If the degree of any vertex is one, then that vertex is called pendent vertex

Directed edges:

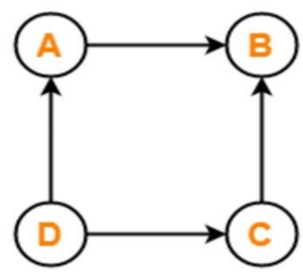
In a graph $G = (V, E)$, on edge which is associated with an ordered pair of $V \times V$ is called a directed edge of G.

Undirected edge:

If an edge which is associated with an unordered pair of nodes is called an undirected edge.

Digraph:

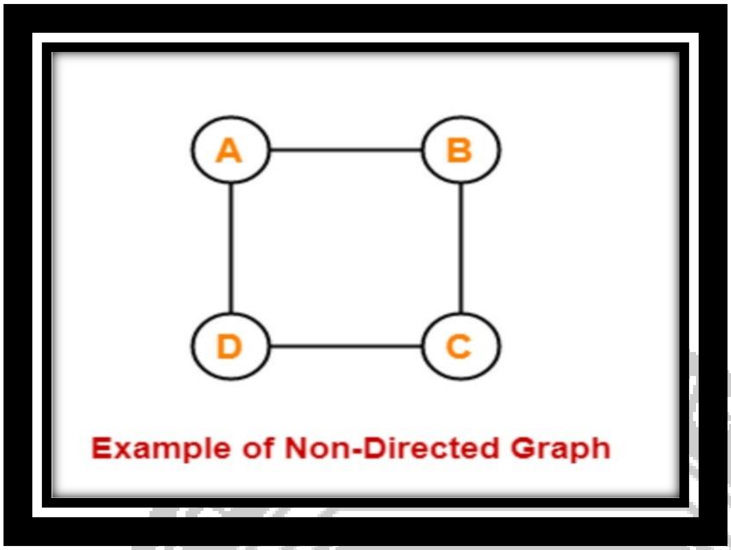
A graph in which every edge is directed edge is called a digraph or directed graph.



Example of Directed Graph

Undirected graph:

A graph in which every edge is undirected is called an undirected graph.

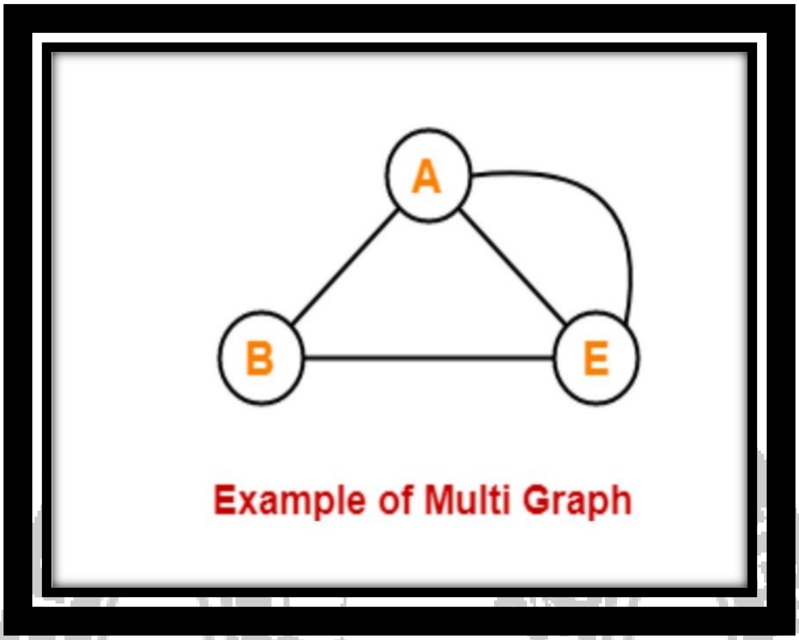


Mixed graph:

If some edges are directed and some are undirected in a graph, the graph is called mixed graph.

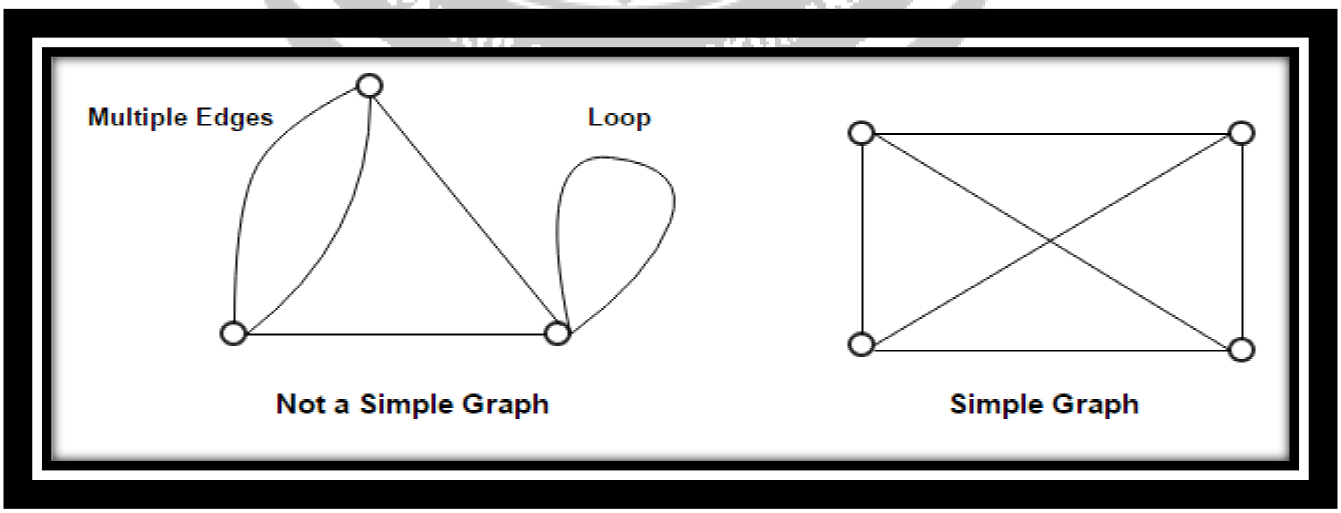
Multigraph:

A graph which contains some parallel edges is called a multigraph.



Pseudograph:

A graph in which loops are parallel edges are allowed is called a Pseudo graph.

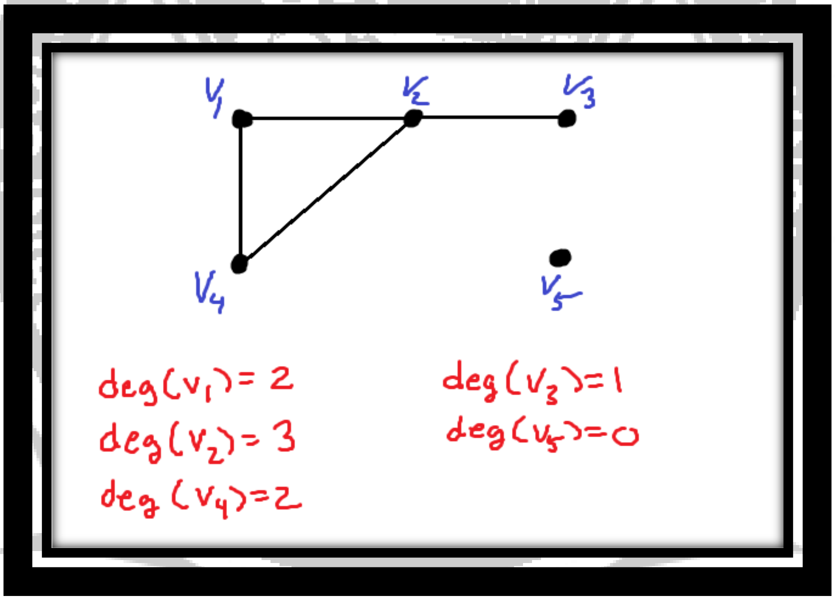


Graph Terminology:

Degree of a vertex:

The number of edges incident at the vertex v_i is called the degree of the vertex with self loops counted twice and it is denoted by $d(v_i)$.

Example:



(i) $d(v_1) = 2$

(ii) $d(v_2) = 3$

(iii) $d(v_3) = 1$

(iv) $d(v_4) = 2$

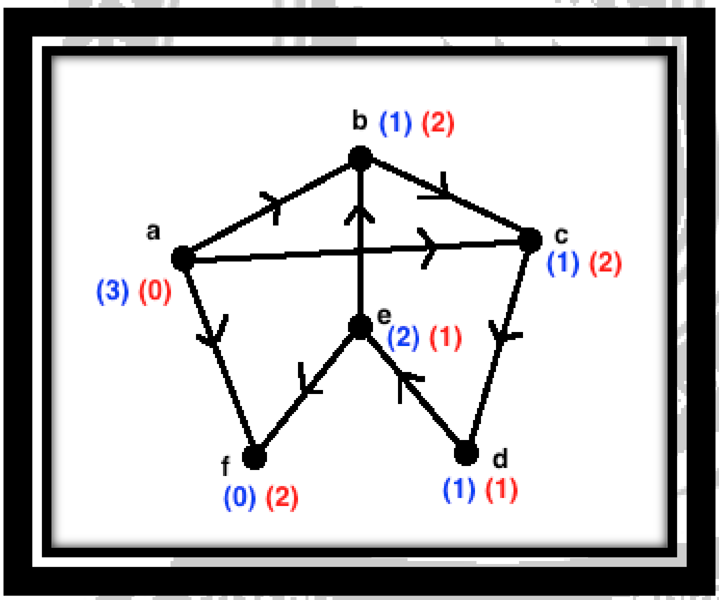
(v) $d(v_5) = 0$

In – degree and out – degree of a directed graph:

In a directed graph, the in – degree of a vertex V, denoted by $deg^{-}(V)$and defined by the number of edges with V as their terminal vertex.

The out – degree of V, denoted by $deg^{+}(V)$, is the number of edges with V as their initial vertex.

Example:



In – degree	Out – degree	Total degree
$deg^{-}(a) = 0$	$deg^{+}(a) = 3$	$deg(a) = 3$
$deg^{-}(b) = 2$	$deg^{+}(b) = 1$	$deg(b) = 3$
$deg^{-}(c) = 2$	$deg^{+}(c) = 1$	$deg(c) = 3$

$deg^-(d) = 1$	$deg^+(d) = 1$	$deg(d) = 2$
$deg^-(e) = 1$	$deg^+(e) = 2$	$deg(e) = 3$
$deg^-(f) = 2$	$deg^+(f) = 0$	$deg(f) = 2$

Note:

A loop at a vertex contributes 1 to both the in – degree and the out – degree of this vertex.

Theorem: 1(The Handshaking Theorem)

Let $G = (V, E)$ be an undirected graph with e edges then $\sum_{v\in V} deg(v) = 2e$.

The sum of degrees of all the vertices of an undirected graph is twice the number of edges of the graph and hence even.

Proof:

Since every edge is incident with exactly two vertices, every edge contributes 2 to the sum of the degree of the vertices.

All the 'e' edges contribute $(2e)$ to the sum of the degrees of vertices.

Hence $\sum_{v\in V} deg(v) = 2e$

Hence the proof.

Theorem: 2

In a undirected graph, the number of odd degree vertices are even.

Proof:

Let V_1 and V_2 be the set of all vertices of even degree and set of all vertices of odd degree, respectively, in a graph $G = (V, E)$.

$\Rightarrow \sum d(v) = \sum_{v_i \in V_1} d(v_i) + \sum_{v_j \in V_2} d(v_j)$

By Handshaking theorem, we have

$$\Rightarrow 2e = \sum_{v_i \in V_1} d(v_i) + \sum_{v_j \in V_2} d(v_j)$$

Since each $\deg(v_i)$ is even, $\sum_{v_i \in V_1} d(v_i)$ is even

As left hand side of equation (1) is even and the first expression of the RHS of (1) is even, we have the second expression on the RHS must be even.

$\sum_{v_j \in V_2} d(v_j)$ is even.

Since each $d(v_j)$ is odd, the number of terms contained in $\sum_{v_j \in V_2} d(v_j)$ is even.

i.e., The number of vertices of odd degree is even.

Hence the proof.

Theorem: 3

The maximum number of edges in a simple graph with "n" vertices is $\frac{n(n-1)}{2}$

Proof:

We prove this theorem by the principle of Mathematical induction.

For $n = 1$, a graph with one vertex has no edges.

The result is true for $n = 1$.

For $n = 2$, a graph with 2 vertices may have atmost one edge.

$\Rightarrow \frac{2(2-1)}{2} = 1$

The result is true for $n = 2$.

Assume that the result is true for $n = k$.

i.e., a graph with k vertices has atmost $\frac{k(k-1)}{2}$ edges.

When $n = k + 1$, let G be a graph having "n" vertices and G' be the graph obtained from G by deleting one vertex say $v \in V(G)$.

Since G' has k vertices, then by the hypothesis G' has atmost $\frac{k(k-1)}{2}$ edges.

Now add the vertex "v" to G'. Such that "v" may be adjacent to all the k vertices of G'.

The total number of edges in G are,

$$\frac{k(k-1)}{2} + k = \frac{k^2-k+2k}{2}$$

$$= \frac{k^2+k}{2}$$

$$= \frac{k(k+1)}{2}$$

$$= \frac{(k+1)(k+1-1)}{2}$$

The result is true for $n = k + 1$

Hence the maximum number of edges in a simple graph with "n" vertices is $\frac{n(n-1)}{2}$.

Hence the proof.

Theorem: 4

If all the vertices of an undirected graph are each of degree k, show that the number of edges of the graph is a multiple of k.

Proof:

Let $2n$ be the number of vertices of the given graph. . . . (1)

Let n_e be the number of edges of the given graph.

By Handshaking theorem, we have $\sum_{i=1}^{2n} \deg V_i = 2\, n_e$

$\Rightarrow 2nk = 2n_e$ using (1)

$\Rightarrow nk = n_e$

$\Rightarrow$ number of edges = multiple of k.

The number of edges of the given graph is a multiple of k.

Example:1

How many edges are there in a graph with ten vertices each of degree six.

Solution:

Let e be the number of edges of the graph.

$\Rightarrow 2e$ = Sum of all degrees

$= 10 X 6 = 60$

$\Rightarrow 2e = 60$

$\Rightarrow e = 30$

There are 30 edges.

Example: 2

Can a simple graph exist with 15 vertices of degree 5.

Solution:

$\Rightarrow 2e = \sum d(v)$

$\Rightarrow 2e = 15X\,5 = 75$

$\Rightarrow e = \frac{75}{2}$

Which is not an integer.

Such a graph does not exist.

(or) By theorem (2) in a graph the number of odd degree vertices is even.

Therefore, it is not possible to have 15 vertices, which is of odd degree.

Such a graph does not exist.

Example: 3

For the following degree sequences 4, 4, 4, 3, 2 find if there exist a graph or not.

Solution:

Sum of the degree of all vertices $= 4 + 4 + 4 + 3 + 2 = 17$

Which is an odd number.

Such a graph does not exist.

Example: 4

Does there exist a simple graph with five vertices of the following degrees? If so draw such graph (a) 1, 1, 1, 1, 1 (b) 3, 3, 3, 3, 2

Solution:

We know that in any graph the number of odd degree vertices is always a even.

In case (a) number of odd degree vertices is 5 (not an even)

Such graph does not exist.

For case (b)

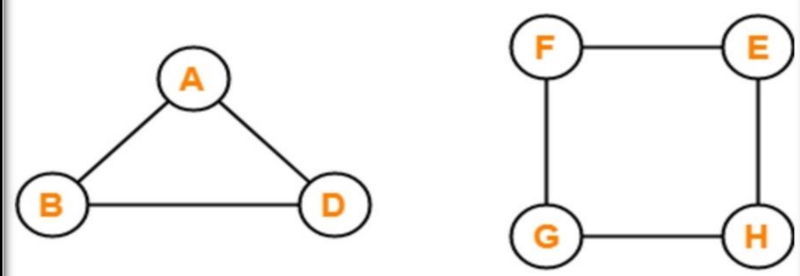
Sum of degree = 14 = even

The graph exist.

Special Types of Graphs

Regular Graph

If every vertex of a simple graph has the same degree, then the graph is called a regular graph.



Examples of Regular Graph

If every vertex in a regular graph has degree k, then the graph is called k – regular.

Complete Graph

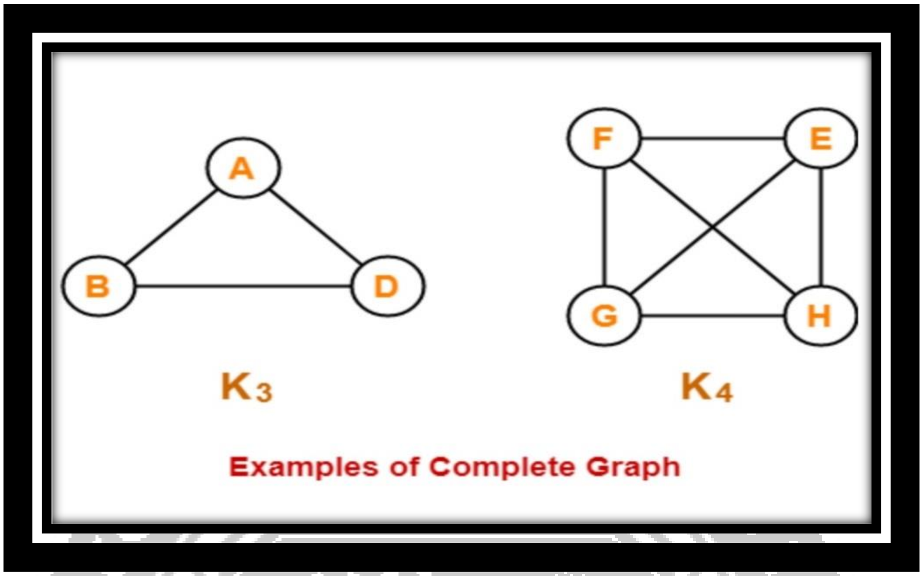
In a graph, if there exist an edge between every pair of vertices, then such a graph is called complete graph.

In a graph if every pair of vertices are adjacent then such a graph is called complete graph.

If is noted that, every complete graph is a regular graph. In fact every complete graph with n vertices is a $(n-1)$ regular graph.

The complete graph on n vertices is denoted by k_n. The graphs k_n for

$n = 1, 2, 3, 4, 5$ are



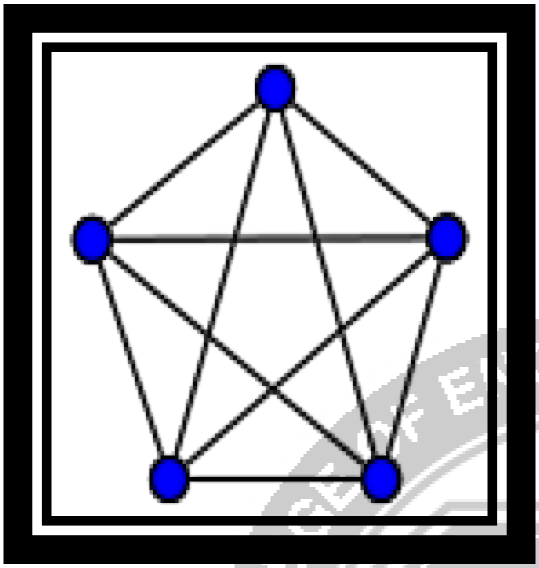
Example: 1

Draw the complete graph k_5 with vertices A, B, C, D, E. Draw all complete sub graph of k_5 with 4 vertices.

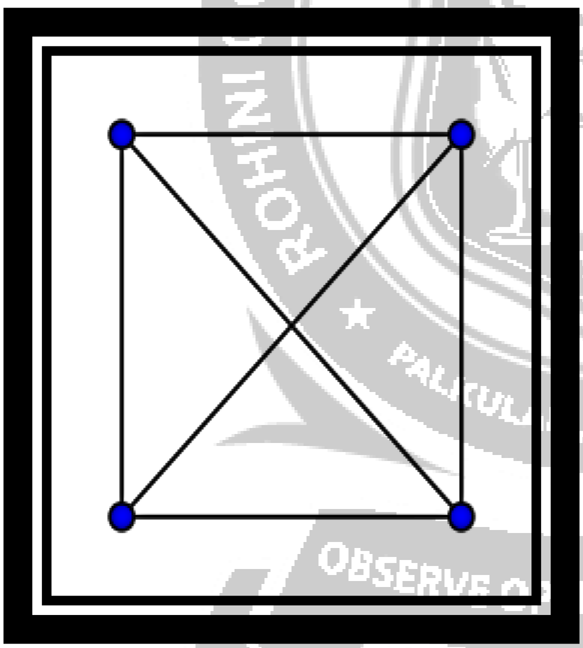
Solution:

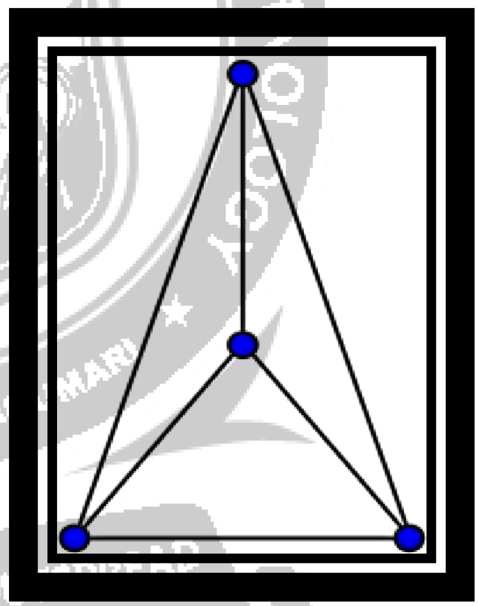
In a graph, if there exist an edge between every pair of vertices, then such a graph is called complete graph.

i.e., In a graph if every pair of vertices are adjacent, then such a graph is called complete graph. Complete graph k_5 is



Now, complete subgraph of k_5 with 4 vertices are





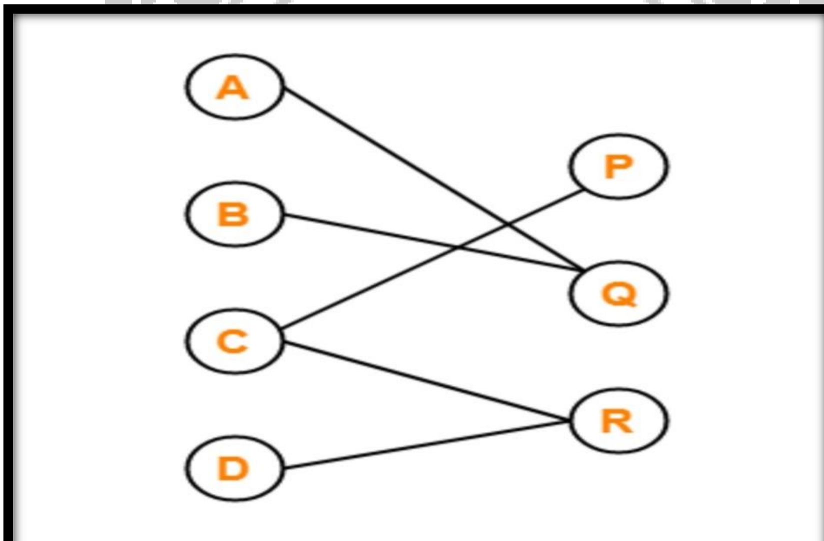
Bipartite Graph

A graph G is said to be bipartite if its vertex set $V(G)$ can be partitioned into two disjoint non empty sets V_1 and V_2, $V_1 \cup V_2 = V(G)$, such that every edge in $E(G)$

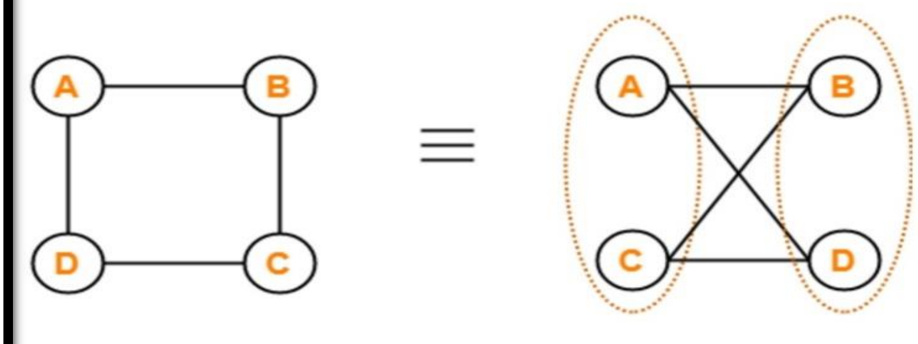
has one end vertex in V_1 and another end vertex in V_2 . (So that no edges in , connects either two vertices in V_1 or two vertices in V_2.)

For example, consider the graph G

Then G is a Bipartite graph.



Example of Bipartite Graph

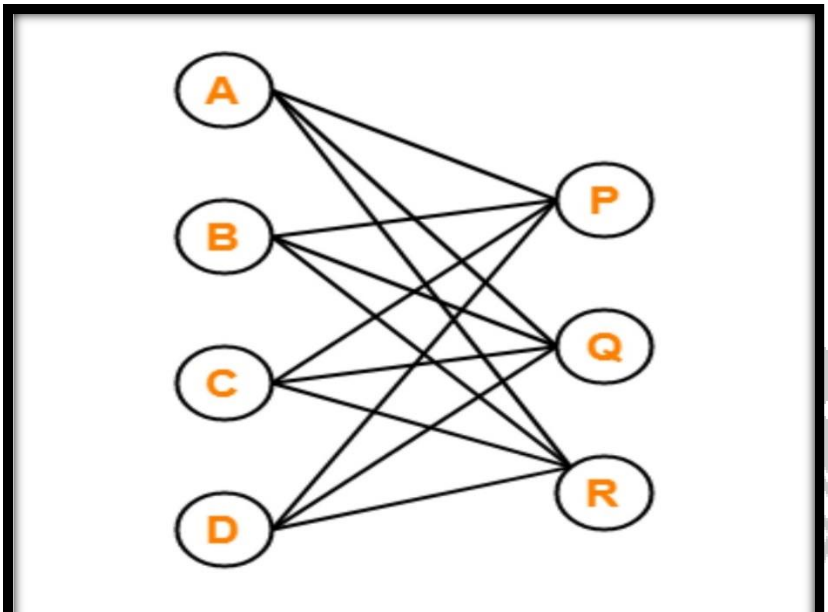


Example of Bipartite Graph

Complete Bipartite Graph:

A bipartite graph G, with the partition V_1 and V_2, is called complete bipartite graph, if every vertex in V_1 is adjacent to every vertex in V_2. Clearly, every vertex in V_2 is adjacent to every vertex in V_1.

A complete bipartite graph with 'm' and 'n' vertices in the bipartition is denoted by $k_{m,n}$.



Example of Complete Bipartite Graph

Subgraph:

A graph $H = (V', E')$ is called a subgraph of $G = (V, E)$, if $V' \subseteq V$ and $E' \subseteq E$.

In other words, a graph H is said to be a subgraph is said to be a subgraph of G, if all the vertices and all the edges of H are in G and if the adjacency is preserved in H exactly as in G.

Hence, we have the following

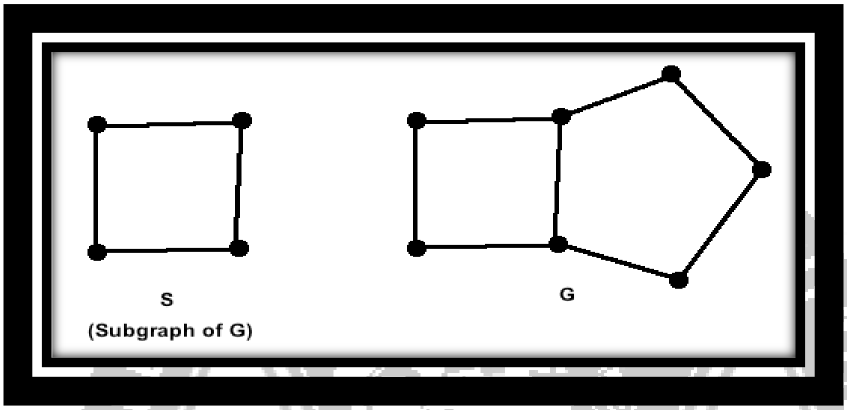
(i)Each graph has its own subgraph.

(ii)A single vertex in a graph G is a subgraph of G.

(iii)A single edge in G, together with its end vertices is also a subgraph of G.

(iv)A subgraph of a subgraph of G is also a subgraph of G.

(v)H is a proper subgraph of G if $H \neq G$.



Graph representation:

Adjacency Matrix of a simple graph:

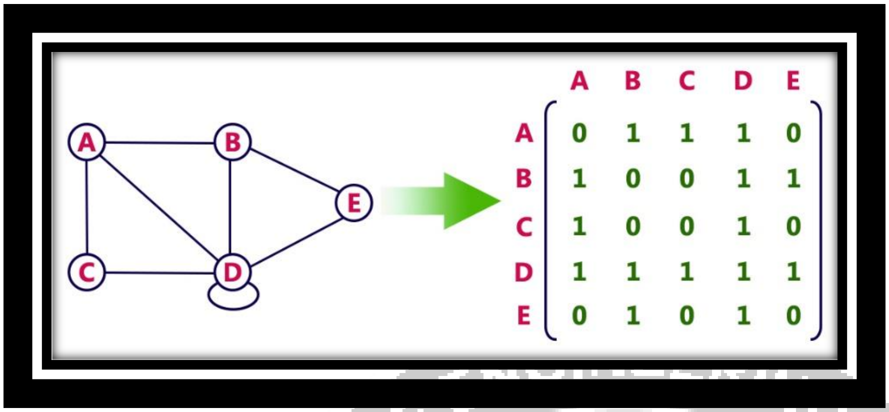
Let $G = (V, E)$ be a simple graph with n – vertices $\{v_1, v_2, \ldots, v_n\}$. Its adjacency matrix is denoted by $A = [a_{ij}]$ and defined by

$$A = [a_{ij}] = \begin{cases} 1, if\ there\ exist\ an\ edge\ between\ v_i\ and\ v_j \\ 0, otherwise \end{cases}$$

Example: 1

Find adjacency matrix of the graph given below.

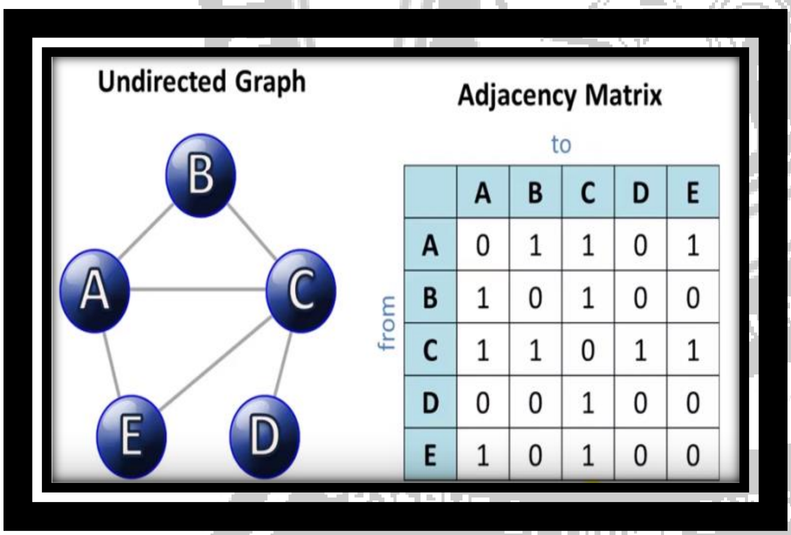
Solution:



Example: 2

Find adjacency matrix of the graph given below.

Solution:



Incidence matrices:

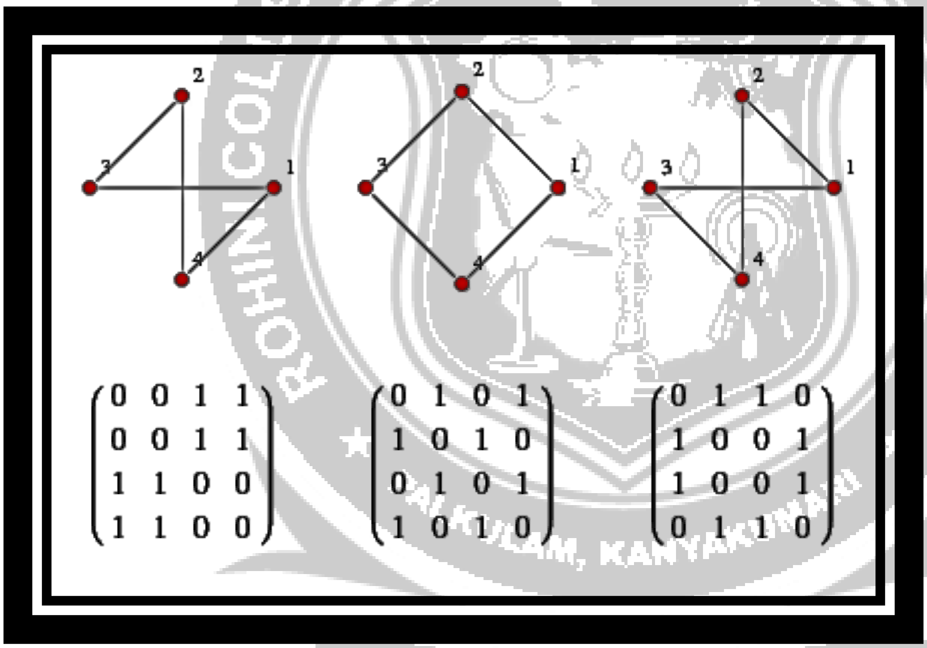
Let $G = (V, E)$ be an undirected graph with n vertices $\{V_1, V_2, \dots, V_n\}$ and m edges $\{e_1, e, \dots, e_m\}$. Then the $(n \times m)$ matrix $B = [b_{ij}]$, where

$$B = [b_{ij}] = \begin{cases} 1, when\ edge\ e_j\ incident\ on\ V_i \\ 0, otherwise \end{cases}$$

Example: 1

Find incidence matrix of the following graph and your observations regarding the entries of B.

Solution:



Path Matrix:

Let $G = (V, E)$ be a simple digraph in which $|V| = n$ and the nodes of g are assumed to be ordered. An $n \times n$ matrix P whose elements are given by

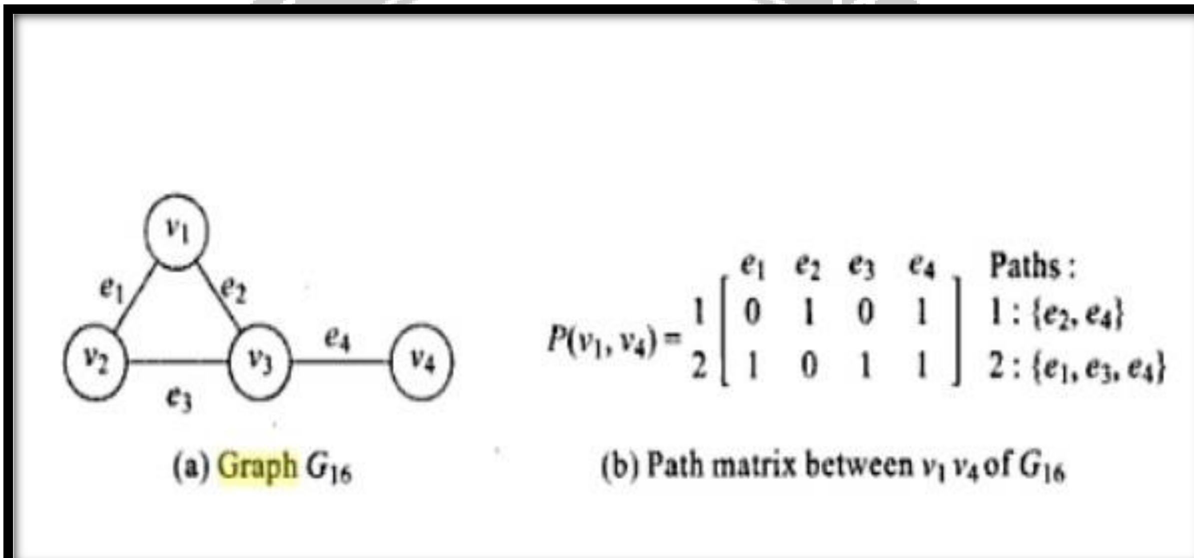
$$P_{ij} = \begin{cases} 1, If\ there\ exists\ a\ path\ from\ V_i\ to\ V_j \\ 0, otherwise \end{cases}$$

is called a path matrix (reachability matrix) of the graph G.

Example: 1

Find path matrix

Solution:



(a) Graph G_{16} (b) Path matrix between v_1 v_4 of G_{16}

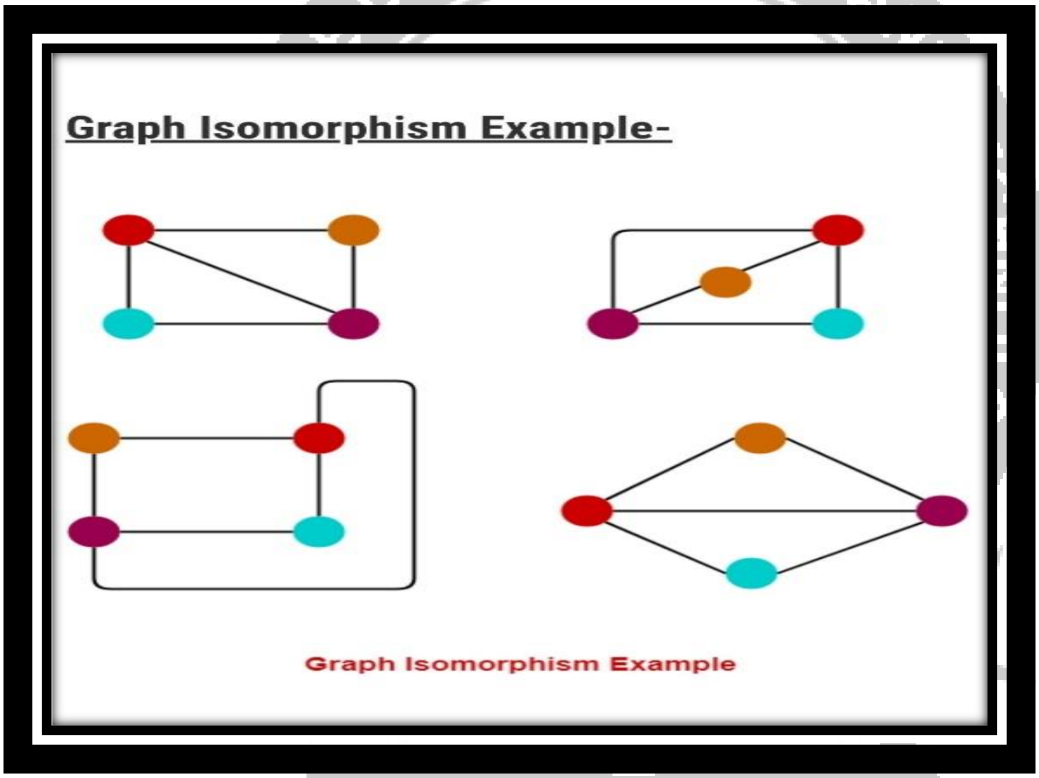
Note:

Path Matrix is very useful in communications and transportation networks.

Graph Isomorphism:

Two graphs G_1 and G_2 are said to be isomorphic to each other, if there exist a one – to –one correspondence between the vertex sets which preserves adjacency of the vertices.

The Graph $G_1 = (V_1, E_1)$ is isomorphic to the graph $G_2 = (V_2, E_2)$ if there is a one – to – one correspondence between the vertex sets V_1 and V_2 and the edge sets E_1 and E_2 in such a way that if e_1 is incident on u_1 and V_1 in G_1, then the corresponding edge e_2 in G_2 is incident on u_2 and V_2 which correspondence is called graph isomorphism.



However, the definition of isomorphism of two graphs were easy, but the given graph having "n" vertices itself has n! ways of one – to – one correspondence.

So, before going to isomorphism, we can verify whether they have the same number of vertices and edges and if the degree sequence of the graphs are same. If not, the we can say the graphs are not isomorphic.

Note:

If G_1 and G_2 are isomorphic then G_1 and G_2 have

(i)The same number of vertices.

(ii)The same number of edges.

(iii)An equal number of vertices with a given degree.

However, these conditions are not sufficient for graph isomorphism.