SAMPLING THEOREM

It is one of useful theorem that applies to digital communication systems.

Sampling theorem states that "A band limited signal $\boldsymbol{x(t)}$ with $\boldsymbol{X(\omega) = 0}$ for $|m| \geq \boldsymbol{\omega m}$ can be represented into and uniquely determined from its samples $\boldsymbol{x(nT)}$ if the sampling frequency $\boldsymbol{fs \geq 2fm}$, where $\boldsymbol{fm}$ is the frequency component present in it".

(i.e) for signal recovery, the sampling frequency must be at least twice the highest frequency present in the signal.

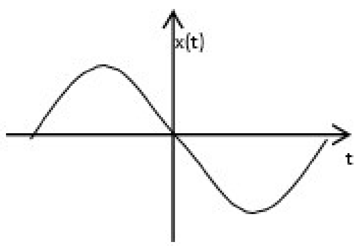

Fig 4.1

$\delta_T(t)$ is impulse train

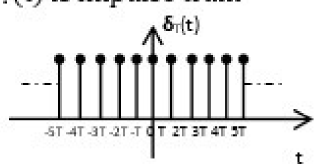

Fig 4.2

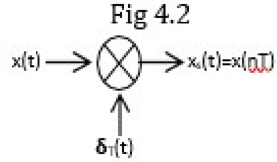

Fig 4.3

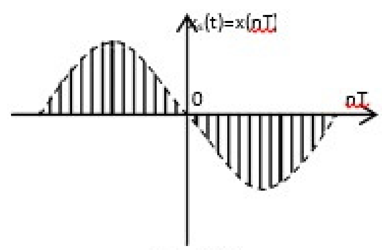

Fig 4.4

Analogsignal$x(t)$is input signal as shown in Fig 4.1, $\delta T(t)$ is the train of impulseshowninFig4.2. Sampled signal $xs(t)$ is the product of signal $x(t)$ and impulsetrain$\delta T(t)$asshown in Fig 4.2

$$\therefore x_s(t) = x(t).\,\delta_T(t)$$

$$we\ know\ \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\,\omega_s t}$$

$$\therefore x_s(t) = x(t).\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\,\omega_s t}$$

Applying Fourier transform on both sides

$$X_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F[x(t)e^{jn\omega_s t}]$$

$$X_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

$$where\ \omega_s = 2\pi f_s = \frac{2\pi}{T}$$

$$\therefore X_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - \frac{2\pi n}{T})$$

$$(or)$$

$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \qquad where\ f_s = \frac{1}{T}$$

Where $X(\omega)$ *or* $X(f)$ *is Spectrum of input signal.*

Where $X_s(\omega)$ *or* $X_s(f)$ *is Specturm of sampled signal.*

Spectrum of continuous time signal x(t) with maximum frequency ωm is shown in Fig 4.5

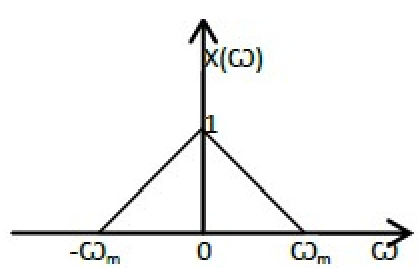

Fig 4.5 Spectrum of x(t)

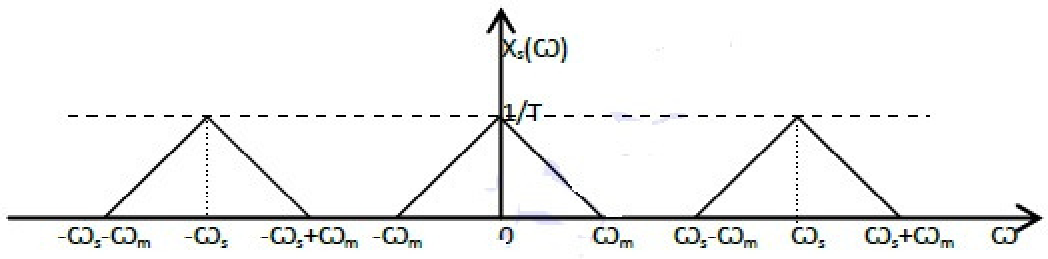

Fig 4.6 Spectrum of $x_s(t)$ when $\omega_s - \omega_m > \omega_m$

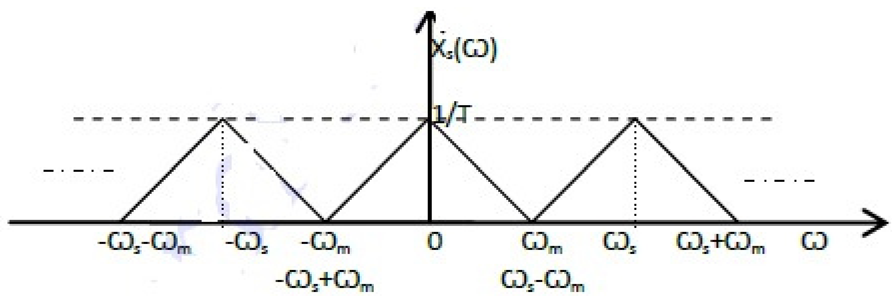

Fig 4.7 Spectrum of $x_s(t)$ when $\omega_s - \omega_m = \omega_m$

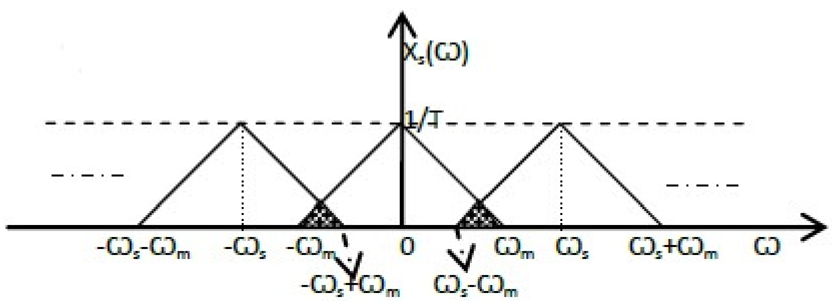

Fig 4.8 Spectrum of $x_s(t)$ when $\omega_s - \omega_m < \omega_m$

For $\omega s > 2\omega m$

The spectral replicates have a larger separation between them known as guard band which makes process of filtering much easier and effective. Even a non-ideal filter which does not have a sharp cut off can also be used.

For $\omega s = 2\omega m$

There is no separation between the spectral replicates so no guard band exists and $X(\omega)$ can be obtained from $Xs(\omega)$ by using only an ideal low pass filter (LPF) with sharp cutoff.

For $\omega_s < 2\omega_m$ The low frequency component in $X_s\ \omega$ overlap on high frequency components of $X\ \omega$ so that there is presence of distortion and $X\ \omega$ cannot be recovered from $X_s\ \omega$ by using any filter. This distortion is called aliasing.

So we can conclude that the frequency spectrum of $X_s(\omega)$ is not overlapped for $\omega_s - \omega_m \geq \omega_m$, therefore the Original signal can be recovered from the sampled signal.

For $\omega_s - \omega_m < \omega_m$, the frequency spectrum will overlap and hence the original signal cannot be recovered from the sampled signal.

∴ For signal recovery,

$$\omega_s - \omega_m \geq \omega_m (i.e)\ \ \boldsymbol{\omega_s \geq 2\omega_m}$$

$$(or)$$

$$\boldsymbol{f_s \geq 2f_m}$$

i.e.,Aliasing can be avoided if $f_s \geq 2f_m$

Aliasingeffect(or)foldover effect

Itisdefinedasthephenomenon in which a high frequency component in the frequencyspectrumofsignal takes identity of a lower frequency component in the spectrumofthesampledsignal.

Whenf_s<2f_m,(i.e) when signal is under sampled, the individual terms in equation

$$X_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} x(\omega - n\omega_s)$$

getoverlap.Thisprocess of spectral overlap is called frequency folding effect.

Occurrence of aliasing

Aliasing Occurs if

i) The signal is not band-Limited to a finite range.

ii) The sampling rate is too low.

To Avoid Aliasing

i) $x(t)$ should be strictly band limited.

It can be ensured by using anti-aliasing filter before the sampler.

ii) fs should be greater than $2fm$.

Nyquist Rate

It is the theoretical minimum sampling rate at which a signal can be sampled and still be reconstructed from its samples without any distortion

$$Nyquist\ rate\ fN = 2fm\ .\ Hz$$

Data Reconstruction or Interpolation

The process of obtaining analog signal $x(t)$ from the sampled signal $xs(t)$ is called data reconstruction or interpolation.

$$we\ know\ x_s(t) = x(t).\delta_T(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

$$\delta(t-nT)\ exist\ \ only\ at\ t = nT$$

$$\therefore x_s(t) = x(nt)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

The reconstruction filter, which is assumed to be linear and time invariant, has unit impulse response h(t).

The reconstruction filter, output $y(t)$ is given by convolution of $xs(t)$ and h(t).

$$\therefore y(t) = x_s(t) * h(t) = \int_{-\infty}^{\infty} x(nT) \sum_{n=-\infty}^{\infty} \delta(\tau - nT).\ h(t-\tau)d\tau$$

$$= \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} \delta(\tau - nT)\ h(t-\tau)d\tau$$

$$\delta(\tau - nT)\ exist\ only\ at\ \tau = nT$$

$$\delta(\tau - nT) = 1\ at\ \tau = nT$$

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT)h(t-nT)$$

Ideal Reconstruction filter

The sampled signal $xs(t)$ is passed through an ideal LPF (Fig 4.9) with bandwidth greater than fm and a pass band amplitude response of T, then the filter output is $x(t)$.

Transfer function of ideal reconstruction filter is

$$H(f) = T\ ;\ \ |f| < 0.5f_s$$

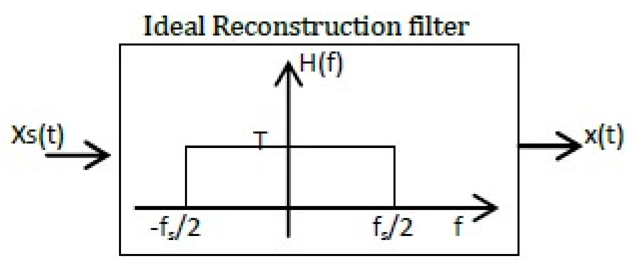

Fig 4.9

The impulse response of ideal reconstruction filter is

$$h(t) = \int_{\frac{-f_s}{2}}^{\frac{f_s}{2}} Te^{j\omega t}\ df$$

$$= \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} T\, e^{j2\pi ft}\, df = T\left[\frac{e^{j2\pi ft}}{j2\pi t}\right]_{-\frac{f_s}{2}}^{\frac{f_s}{2}} = \frac{T}{j2\pi t}\left[e^{j2\pi\frac{f_s}{2}t} - e^{-j2\pi\frac{f_s}{2}t}\right]$$

$$= \frac{1}{f_s\pi t}\left[\frac{e^{j2\pi\frac{f_s}{2}t} - e^{-j2\pi\frac{f_s}{2}t}}{2j}\right] = \frac{1}{\pi f_s t}\sin \pi f_s t = \sin c\, \pi f_s t$$

$$\therefore h(t - nT) = \sin c\, \pi f_s (t - nT) \quad \ldots\ldots\ldots \langle 1 \rangle$$

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT) h(t - nT)$$

Substitute equation 1 in above equation

$$\therefore y(t) = \sum_{n=-\infty}^{\infty} x(nT)\ \sin c\, \pi f_s (t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\ \sin c\, \pi \left(\frac{t}{T} - n\right)$$

$$\left[\because f_s = \frac{1}{T}\right]$$