TAYLORS AND LAURENTS SERIES

In this section, we find a power series for the given analytic function. Taylor's series is a series of positive powers while Laurent's series is a series of both positive and negative powers.

Taylor's Series

If $f\ (z)$ is analytic inside and on a circle C with centre at point 'a' and radius 'R' then at each point Z inside C,

$$f\ (z) = f(a) + (z-a)\frac{f'(a)}{1!} + (z-a)^2\frac{f''(a)}{2!} + \cdots$$

(OR)

$$f(z) = \sum_{n=0}^{\infty}\frac{(z-a)^n}{n!}f^n(a)$$

This is known as Taylor's series of $f(z)$ about $z\ = a$.

Note: 1 Putting $a = 0$ in the Taylor's series we get

$f\ (z) = f(0) + (z-0)\frac{f'(0)}{1!} + (z-0)^2\frac{f''(0)}{1!} + \cdots$ this series is called Maclaurin's Series.

Note: 2 The Maclaurin's for some elementary functions are

1) $(1-z)^{-1} = 1 + z + z^2 + z^3\ + \cdots$, when $|z| < 1$
2) $(1+z)^{-1} = 1 - z + z^2 - z^3\ + \cdots$, when $|z| < 1$
3) $(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3\ + \cdots$, when $|z| < 1$
4) $(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3\ + \cdots$, when $|z| < 1$
5) $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots$ when $|z| < \infty$
6) $e^z = 1 - \frac{z}{1!} + \frac{z^2}{2!} + \cdots$ when $|z| < \infty$
7) $\sin z =\ z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$ when $|z| < \infty$
8) $\cos z =\ 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$ when $|z| < \infty$

LAURENTS SERIES

If $c_1\ and\ c_2$ are two concentric circles with centre at $z = a$ and radii $r_1\ and\ r_2\ (r_1 < r_2)$ and if $f(z)$ is analytic inside on the circles and within the annulus between $c_1\ and\ c_2$ then for any z in the annulus, we have

$$f(z) =\ \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}\ \ldots (1)$$

Where $a_n = \frac{1}{2\pi i}\int_{c_1}\frac{f(z)}{(z-a)^{n+1}}\,dz$ and $b_n = \frac{1}{2\pi i}\int_{c_2}\frac{f(z)}{(z-a)^{1-n}}\,dz$ and the integration being taken in positive direction. This series (1) is called Laurent series of $f(z)$ about the point $z = a$

Note:

1) If $f(z)$ is analytic inside c_2, then the Laurent's series reduces to the Taylor series of $f(z)$with centre a,since the negative powers in Laurent's series is Zero.
2) As the Taylor's and Laurent's expansion in the regions are unique, they can find by simpler method such as binomial series.

3) In Laurent's series the part $\sum_{n=0}^{\infty} a_n(z-a)^n$, consisting of positive powers of $(z-a)$ is called the analytic part of Laurent's series, while $\sum_{n=1}^{\infty} b_n(z-a)^{-n}$consisting of negative powers of $(z-a)$ is called the principal part of Laurent's series.

4) The coefficient of $\frac{1}{z-a}$ $(i.e)\ b$, in the Laurent's expansion of $f(z)$ about a singularity $z = a$ valid in region $0 < |z-a| < r$ is also called residue.

$(i.e)$ coeff of $\frac{1}{z-a} = Res\ [f(z), z = a]$

Problems based on Taylor's series

Example: 4.18 Expand $f(z) = cosz$ as a Taylor's series about $z = \frac{\pi}{4}$.

Solution:

Function	Value of function at $z = \frac{\pi}{4}$
$f(z) = cosz$	$f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$
$f'(z) = -sinz$	$f'\left(\frac{\pi}{4}\right) = -sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
$f''(z) = -cosz$	$f''\left(\frac{\pi}{4}\right) = -cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
$f'''(z) = sinz$	$f''\left(\frac{\pi}{4}\right) = sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

The Taylor series of $f(z)$ about $z = \frac{\pi}{4}$ is $f\ (z) = f\left(\frac{\pi}{4}\right) + \left(z-\frac{\pi}{4}\right)\frac{f'\left(\frac{\pi}{4}\right)}{1!} + \left(z-\frac{\pi}{4}\right)^2\frac{f''\left(\frac{\pi}{4}\right)}{2!} + \cdots$

$$cosz = \frac{1}{\sqrt{2}} + \left(z-\frac{\pi}{4}\right)\frac{-\frac{1}{\sqrt{2}}}{1!} + \left(z-\frac{\pi}{4}\right)^2\frac{-\frac{1}{\sqrt{2}}}{2!} + \cdots$$

Example: 4.19 Expand $f(z) = log\ (1+z)$ as a Taylor's series about $z = 0$.

Solution:

Function	Value of function at $z = 0$
$f(z) = log\ (1+z)$	$f(0) = log\ (1+0) = 0$
$f'(z) = \frac{1}{1+z}$	$f'(0) = \frac{1}{1+0} = 1$
$f''(z) = \frac{-1}{(1+z)^2}$	$f''(0) = \frac{-1}{(1+0)^2} = -1$
$f'''(z) = \frac{2}{(1+z)^3}$	$f'''(0) = \frac{2}{(1+0)^3} = 2$

The Taylor series of $f(z)$ about $z = 0$ is

$$f\ (z) = f(0) + (z-0)\frac{f'(0)}{1!} + (z-0)^2\frac{f''(0)}{2!} + \cdots$$

$$log\ (1+z) = 0 + (z)\frac{1}{1!} + (z)^2\frac{-1}{2!} + \cdots$$

$$log\ (1+z) = (z)\frac{1}{1!} - (z)^2\frac{1}{2!} + \cdots$$

Example: 4.20 Expand $f(z) = \frac{1}{z-2}$as a Taylor's series about $z = 1$.

Solution:

Function	Value of function at $z = 1$
$f(z) = \frac{1}{z-2}$	$f(z) = \frac{1}{1-2} = -1$
$f'(z) = \frac{-1}{(z-2)^2}$	$f'(1) = \frac{-1}{(1-2)^2} = -1$
$f''(z) = \frac{2}{(z-2)^3}$	$f''(1) = \frac{2}{(1-2)^3} = -2$
$f'''(z) = \frac{-6}{(z-2)^4}$	$f'''(0) = \frac{-6}{(z-2)^4} = -6$

The Taylor series of $f(z)$ about $z = 1$ is

$$f(z) = f(1) + (z-1)\frac{f'(1)}{1!} + (z-1)^2\frac{f''(1)}{2!} + \cdots$$

$$\frac{1}{z-2} = -1 + (z-1)\frac{-1}{1!} + (z-1)^2\frac{-2}{2!} + \cdots$$

Problems based on Laurent's Series

Working rule to expand $f(z)$ as a Laurent's Series

Let $f(z) = \frac{1}{z+a} + \frac{1}{z+b}$ $with\ a < b$

(i) To expand $f(z)$ in $|z| < a$, rewrite $f(z)$as

$$f(z) = \frac{1}{a(1+z/a)} + \frac{1}{b(1+z/b)}$$

$$= \frac{1}{a}(1+z/a)^{-1} + \frac{1}{b}\left(1+z/b\right)^{-1}$$

Now use Binomial expansion.

(ii)To expand $f(z)$ in $|z| > a$, rewrite $f(z)$ as

$$f(z) = \frac{1}{z(1+a/z)} + \frac{1}{z(1+b/z)}$$

$$= \frac{1}{z}(1+a/z)^{-1} + \frac{1}{z}\left(1+b/z\right)^{-1}$$

Now use Binomial expansion.

(iii)To expand $f(z)$ in $a < |z| < b$, rewrite $f(z)$as

$$f(z) = \frac{1}{z(1+a/z)} + \frac{1}{b(1+z/b)}$$

$$= \frac{1}{z}(1+a/z)^{-1} + \frac{1}{b}\left(1+z/b\right)^{-1}$$

Now use Binomial expansion.

Example: 4.21 Expand $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ as a Laurent's series if (i) $|z| < 2$ (ii) $|z| > 3$

(iii) $2 < |z| < 3$

Solution:

Given $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ is an improper fraction. Since degree of numerator and degree of denominator of $f(z)$ are same

$\therefore$ Apply division process

$$\begin{array}{r|l} & 1 \\ \hline z^2+5z+6 & z^2-1 \\ & z^2+5z+6 \\ \hline & -5z-7 \end{array}$$

$\therefore \frac{z^2-1}{(z+2)(z+3)} = 1 - \frac{5z+7}{(z+2)(z+3)} \ \dots (1)$

Consider $\frac{5z+7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$

$\Rightarrow 5z+7 = A(z+3) + B(z+2)$

Put $z = -2$, we get $-10+7 = A\,(1)$

$\Rightarrow A = -3$

Put $z = -3$, we get $-15+7 = B(-1)$

$\Rightarrow B = 8$

$$\therefore \frac{5z+7}{(z+2)(z+3)} = \frac{-3}{z+2} + \frac{8}{z+3}$$

$$\therefore (1) \Rightarrow 1 - \frac{3}{z+2} - \frac{8}{z+3}$$

(i) Given $|z| < 2$

$$\therefore f(z) = 1 + \frac{3}{2(1+{}^{z}/_{2})} - \frac{8}{3(1+{}^{z}/_{3})}$$

$$= 1 + \frac{3}{2}\left(1 + {}^{z}/_{2}\right)^{-1} - \frac{8}{3}\left(1 + {}^{z}/_{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \left[\frac{z}{2}\right]^2 + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left[\frac{z}{3}\right]^2 + \cdots\right]$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left[\frac{z}{2}\right]^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left[\frac{z}{3}\right]^n$$

(ii) Given $|z| > 3$

$$\therefore f(z) = 1 + \frac{3}{z(1+{}^{2}/_{z})} - \frac{8}{z(1+{}^{3}/_{z})}$$

$$= 1 + \frac{3}{z}\left(1 + {}^{2}/_{z}\right)^{-1} - \frac{8}{z}\left(1 + {}^{3}/_{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left[\frac{2}{z}\right]^2 + \cdots\right] - \frac{8}{z}\left[1 - \frac{3}{z} + \left[\frac{3}{z}\right]^2 \dots\right]$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left[\frac{2}{z}\right]^n - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left[\frac{3}{z}\right]^n$$

(iii) Given $2 < |z| < 3$

$$\therefore f(z) = 1 + \frac{3}{z(1+{}^{2}/_{z})} - \frac{8}{3(1+{}^{z}/_{3})}$$

$$= 1 + \frac{3}{z}\left(1 + {}^{2}/_{z}\right)^{-1} - \frac{8}{3}\left(1 + {}^{z}/_{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left[\frac{2}{z}\right]^2 + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left[\frac{z}{3}\right]^2 \dots\right]$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left[\frac{2}{z}\right]^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left[\frac{z}{3}\right]^n$$

Example: 4.22 Find the Laurent's series expansion of $f(z) = \frac{7z-2}{z(z-2)(z+1)}$ in $1 < |z+1| < 3$.

Also find the residue of $f(a)$ at $z = -1$

Solution:

Given $f(z) = \frac{7z-2}{z(z-2)(z+1)}$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$$7z - 2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$$

Put $z = 2$, we get $14 - 2 = B\,(2)\,(2+1)$

$$\Rightarrow 12 = 6B$$

$$\Rightarrow B = 2$$

Put $z = -1$, we get $-7 - 2 = C(-1)(-1-2)$

$$\Rightarrow -9 = 3C$$

$$\Rightarrow C = -3$$

Put $z = 0$we get $-2 = A(-2)$

$$\Rightarrow A = 1$$

$$\therefore f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

Given region is $1 < |z+1| < 3$

Let $u = z + 1 \Rightarrow z = u - 1$

$(i.e)\ 1 < |u| < 3$

Now $f(z) = \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u}$

$$= \frac{1}{u(1 - {}^{1}/_{u})} + \frac{2}{-3(1 - {}^{u}/_{3})} - \frac{3}{u}$$

$$= \frac{1}{u}\left(1 - {}^{1}/_{u}\right)^{-1} - \frac{2}{3}\left(1 - {}^{u}/_{3}\right)^{-1} - \frac{3}{u}$$

$$= \frac{1}{u}\left[1 + \frac{1}{u} + \left[\frac{1}{u}\right]^2 + \cdots\right] - \frac{2}{3}\left[1 + \frac{u}{3} + \left[\frac{u}{3}\right]^2 + \cdots\right] - \frac{3}{u}$$

$$= \frac{1}{z+1}\left[1 + \frac{1}{z+1} + \left[\frac{1}{z+1}\right]^2 + \cdots\right] - \frac{2}{3}\left[1 + \frac{z+1}{3} + \left[\frac{z+1}{3}\right]^2 + \cdots\right] - \frac{3}{z+1}$$

$$= \frac{1}{z+1}\sum_{n=0}^{\infty}\left[\frac{1}{z+1}\right]^n - \frac{2}{3}\sum_{n=0}^{\infty}\left[\frac{1}{\frac{z+1}{3}}\right]^n - \frac{3}{z+1}$$

Also $Res[f(z), z = -1]$ =coefficient of $\frac{1}{z+1} = -2$

Example: 4.23 Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in a Laurent's series valid in the region

(i)$|z-1| > 1$(ii) $0 < |z-2| < 1$ (iii) $|z| > 2$ (iv) $0 < |z-1| < 1$

Solution:

Given $f(z) = \frac{1}{(z-1)(z-2)}$

Consider $\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$$\Rightarrow 1 = A(z-2) + B(z-1)$$

Put $z = 2$, we get $1 = B(1)$

$$\Rightarrow B = 1$$

Put $z = 1$we get $1 = A(1-2)$

$$\Rightarrow A = -1$$

$$\therefore f(z) = \frac{-1}{z-1} + \frac{1}{z-2}$$

(i) Given region is $|z-1| > 1$

Let $u = z - 1 \Rightarrow z = u + 1$

$$(i.e)\ |u| > 1$$

Now $f(z) = -\frac{1}{u} + \frac{1}{u-1}$

$$= \frac{-1}{u} + \frac{1}{u(1-{}^{1}/_{u})}$$

$$= \frac{-1}{u} + \frac{1}{u}\left(1 - {}^{1}/_{u}\right)^{-1}$$

$$= \frac{-1}{u} + \frac{1}{u}\left[1 + \frac{1}{u} + \left[\frac{1}{u}\right]^2 + \cdots\right]$$

$$= \frac{-1}{z+1} + \frac{1}{z+1}\left[1 + \frac{1}{z+1} + \left[\frac{1}{z+1}\right]^2 + \cdots\right]$$

$$= \frac{-1}{z+1} + \frac{1}{z+1}\sum_{n=0}^{\infty}\left[\frac{1}{z+1}\right]^n$$

(ii) Given $0 < |z-2| < 1$

Let $u = z - 2 \Rightarrow z = u + 2$

$$(i.e)\ 0 < |u| < 1$$

Now $f(z) = -\frac{1}{u+1} + \frac{1}{u}$

$$= -(1+u)^{-1} + \frac{1}{u}$$

$$= -[1 - u + [u]^2 + \cdots] + \frac{1}{u}$$

$$= -[1 - (z-2) + [z-2]^2 + \cdots] + \frac{1}{z-2}$$

$$= -\sum_{n=0}^{\infty}[-1]^n[z-2]^n + \frac{1}{z-2}$$

(iii) Given $|z| > 2$

Now $f(z) = -\frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{z\left(1-\frac{2}{z}\right)}$

$$= -\frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} + \frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1}$$

$$= -\frac{1}{z}\left[1 + \frac{1}{z} + \left[\frac{1}{z}\right]^2 + \cdots\right] + \frac{1}{z}\left[1 + \frac{2}{z} + \left[\frac{2}{z}\right]^2 + \cdots\right]$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty}\left[\frac{1}{z}\right]^n + \frac{1}{z}\sum_{n=0}^{\to\infty}\left[\frac{2}{z}\right]^n$$

(iv) Given $0 < |z-1| < 1$

Let $u = z - 1 \Rightarrow z = u + 1$

$$(i.e)\ 0 < |u| < 1$$

Now $f(z) = -\frac{1}{u} + \frac{1}{u-1}$

$$= -\frac{1}{u} + \frac{1}{-1\,[1-u]}$$

$$= -\frac{1}{u} - (1-u)^{-1}$$

$$= -\frac{1}{u} - [1 + u + [u]^2 + \cdots]$$

$$= -\frac{1}{z-1} - [1 + z - 1 + [z-1]^2 + \cdots]$$

$$= -\frac{1}{z-1} - \sum_{n=0}^{\infty} [z-1]^n$$

Example: 4.24 Expand $f(z) = \frac{z}{(z+1)(z-2)}$ in a Laurent's series about (i) $z = -1$ (ii) $z = 2$

Solution:

Consider $\frac{z}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2}$

$$\Rightarrow z = A(z-2) + B(z+1)$$

Put $z = 2$, we get $2 = B(3)$

$$\Rightarrow B = \frac{2}{3}$$

Put $z = -1$we get $-1 = A(-3)$

$$\Rightarrow A = \frac{1}{3}$$

$$\therefore f(z) = \frac{1}{3(z+1)} + \frac{2}{3(z-2)}$$

(i)To expand $f(z)$ about $z = -1$

(or) $|z-1| < 1$

Put $z + 1 = u \Rightarrow z = u - 1$

$$\Rightarrow |z-1| < 1 \Rightarrow |u| < 1$$

Now $f(z) = \frac{1}{3u} + \frac{2}{3(u-3)}$

$$= \frac{1}{3u} + \frac{2}{3\left((-3)(1-{}^{u}/_{3})\right)}$$

$$= \frac{1}{3u} - \frac{2}{9}\left(1 - {}^{u}/_{3}\right)^{-1}$$

$$= \frac{1}{3u} - \frac{2}{9}\left[1 + \frac{u}{3} + \left[\frac{u}{3}\right]^2 + \cdots\right]$$

$$= \frac{1}{3(z+1)} - \frac{2}{9}\left[1 + \frac{(z+1)}{3} + \left[\frac{(z+1)}{3}\right]^2 + \cdots\right]$$

$$= \frac{1}{3(z+1)} - \frac{2}{9}\sum_{n=0}^{\infty}\left[\frac{(z+1)}{3}\right]^n$$

(ii) To expand $f(z)$ about $z = 2$

(or) $|z-2| < 1$

Put $z - 2 = u \Rightarrow z = u + 2$

$$\Rightarrow |z-2| < 1 \Rightarrow |u| < 1$$

Now $f(z) = \frac{1}{3(u+3)} + \frac{2}{3(u)}$

$$= \frac{1}{3\left((3)(1+{}^{u}/_{3})\right)} + \frac{2}{3(u)}$$

$$= \frac{1}{9}\left(1 + {}^{u}/_{3}\right)^{-1} + \frac{2}{3(u)}$$

$$= \frac{1}{9}\left[1 - \frac{u}{3} + \left[\frac{u}{3}\right]^2 + \cdots\right] + \frac{2}{3(u)}$$

$$= \frac{1}{9}\left[1 - \frac{(z-2)}{3} + \left[\frac{(z-2)}{3}\right]^2 + \cdots\right] + \frac{2}{3(z-2)}$$

$$= \frac{1}{9}\sum_{n=0}^{\infty}(-1)^n\left[\frac{(z-2)}{3}\right]^n + \frac{2}{3(z-2)}$$

Example: 4.25 Expand the Laurent's series about for $f(z) = \frac{6z+5}{z(z-2)(z+1)}$ in the region $1 < |z+1| < 3$

Solution:

Consider $\frac{6z+5}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$

$$\Rightarrow 6z + 5 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$$

Put $z = 0$, we get $5 = A(-2)(1)$

$$\Rightarrow A = \frac{-5}{2}$$

Put $z = -1$ we get $-11 = C(-1)(-3)$

$$\Rightarrow C = -\frac{11}{3}$$

Put $z = 2$ we get $17 = B(2)(3)$

$$\Rightarrow B = \frac{17}{6}$$

$$\therefore f(z) = \frac{-5}{2z} + \frac{17}{6(z-2)} - \frac{11}{3(z+1)}$$

Given region $1 < |z+1| < 3$

Put $z + 1 = u \Rightarrow z = u - 1$

$(i.e)\ 1 < |u| < 3$

Now $f(z) = \frac{-5}{2(u-1)} + \frac{17}{6(u-3)} - \frac{11}{3u}$

$$= \frac{-5}{2u(1-\frac{1}{u})} + \frac{17}{6(-3)(1-\frac{u}{3})} - \frac{11}{3u}$$

$$= \frac{-5}{2u}\left[1 - \frac{1}{u}\right]^{-1} - \frac{17}{18}\left[1 - \frac{u}{3}\right]^{-1} - \frac{11}{3u}$$

$$= \frac{-5}{2u}\left[1 + \frac{1}{u} + \left[\frac{1}{u}\right]^2 + \cdots\right] - \frac{17}{18}\left[1 + \frac{u}{3} + \left[\frac{u}{3}\right]^2 + \cdots\right] - \frac{11}{3u}$$

$$= \frac{-5}{2(z+1)}\left[1 + \frac{1}{(z+1)} + \left[\frac{1}{(z+1)}\right]^2 + \cdots\right] - \frac{17}{18}\left[1 + \frac{(z+1)}{3} + \left[\frac{(z+1)}{3}\right]^2 + \cdots\right] - \frac{11}{3(z+1)}$$

$$= \frac{-5}{2(z+1)}\sum_{n=0}^{\infty}\left[\frac{1}{(z+1)}\right]^n - \frac{17}{18}\sum_{n=0}^{\infty}\left[\frac{(z+1)}{3}\right]^n - \frac{11}{3(z+1)}$$

Example: 4.26 Find the Laurent's series which represents the function $\frac{z}{(z+1)(z+2)}$ in (i) $|z| < 1$

(ii) $1<|z| < 2$ (iii) $|z| > 2$

Solution:

Consider $\frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$

$$\Rightarrow z = A(z+2)(z+1)$$

Put $z = -2$we get $-2 = B(-1)$

$$\Rightarrow B = 2$$

Put $z = -1$ we get $-1 = A(1)$

$$\Rightarrow A = -1$$

$$\therefore f(z) = \frac{-1}{z+1} + \frac{2}{z+2}$$

(i) Given region $|z| < 1$

$$f(z) = \frac{-1}{z+1} + \frac{2}{2(1+{}^{z}/_{2})}$$

$$= -(1+z)^{-1} + \left(1 + {}^{z}/_{2}\right)^{-1}$$

$$= -[1 - z + z^2 - \cdots] + \left[1 - \frac{z}{2} + \left[\frac{z}{2}\right]^2 - \cdots\right]$$

$$= (-1)\sum_{n=0}^{\infty}(-1)^n z^n + \sum_{n=0}^{\infty}(-1)^n \left[\frac{z}{2}\right]^n$$

(ii) Given region $1 < |z| < 2$

$$f(z) = \frac{-1}{z(1+{}^{1}/_{z})} + \frac{2}{2(1+{}^{z}/_{2})}$$

$$= -{}^{1}/_{z}\left(1 + {}^{1}/_{z}\right)^{-1} + \left(1 + {}^{z}/_{2}\right)^{-1}$$

$$= -{}^{1}/_{z}\left[1 - {}^{1}/_{z} + ({}^{1}/_{z})^2 - \cdots\right] + \left[1 - \frac{z}{2} + \left[\frac{z}{2}\right]^2 - \cdots\right]$$

$$= \left(-{}^{1}/_{z}\right)\sum_{n=0}^{\infty}(-1)^n({}^{1}/_{z})^n + \sum_{n=0}^{\infty}(-1)^n \left[\frac{z}{2}\right]^n$$

(iii) Given region $|z| > 2$

$$f(z) = \frac{-1}{z(1+{}^{1}/_{z})} + \frac{2}{z(1+{}^{2}/_{z})}$$

$$= -{}^{1}/_{z}\left(1 + {}^{1}/_{z}\right)^{-1} + \frac{2}{z}\left(1 + {}^{2}/_{z}\right)^{-1}$$

$$= -{}^{1}/_{z}\left[1 - {}^{1}/_{z} + ({}^{1}/_{z})^2 - \cdots\right] + \frac{2}{z}\left[1 - \frac{2}{z} + \left[\frac{2}{z}\right]^2 - \cdots\right]$$

$$= \left(-{}^{1}/_{z}\right)\sum_{n=0}^{\infty}(-1)^n({}^{1}/_{z})^n + \frac{2}{z}\sum_{n=0}^{\infty}(-1)^n \left[\frac{2}{z}\right]^n$$

(iv) Given region $|z+1| < 1$

Put $z + 1 = u \Rightarrow z = u - 1$

$\therefore\ |z+1| < 1 \Rightarrow |u| < 1$

$$f(z) = \frac{-1}{u} + \frac{2}{u+1}$$

$$= \frac{-1}{u} + 2(1+u)^{-1}$$

$$= \frac{-1}{u} + 2[1 - u + u^2 - \cdots]$$

$$= \frac{-1}{z+1} + 2[1 - (z+1) + ((z+1)^2) - \cdots]$$

$$= \frac{-1}{z+1} + 2\ \sum_{n=0}^{\infty}(-1)^n(z+1)^n$$

EXERCISE: 4.2

(1) Expand $f(z) = e^z$ in a Taylor's series about $z = 0$. **Ans:** $f(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots$

(2) Expand $f(z) = sinz$ in a Taylor's series about $z = \frac{\pi}{4}$. **Ans:** $f(z) = \frac{1}{\sqrt{2}}\left[1 + \frac{\left[z-\frac{\pi}{4}\right]}{1!} - \frac{\left[\left[z-\frac{\pi}{4}\right]\right]^2}{2!} + \cdots\right]$

(3) Find the Laurent's series expansion of $f(z) = \frac{z-1}{(z+2)(z+3)}$valid in the region$2 < |z| < 3$

Ans: $f(z) = \frac{-3}{2}\sum_{n=0}^{\infty}(-1)^n \left[\frac{2}{z}\right]^n + \frac{4}{3}\sum_{n=0}^{\infty}(-1)^n \left[\frac{z}{3}\right]^n$

(4) Expand the Laurent's series the function$f(z) = \frac{3z-7`}{(z-1)(z-2)}$ in the region$1 < |z-1| < 2$

Ans: $f(z) = \frac{-2}{z-1}\sum_{n=0}^{\infty}\left(\frac{1}{z-1}\right)^n - 2\sum_{n=0}^{\infty}\left[\frac{z-1}{2}\right]^n$

(5) Find the Laurent's series the function$f(z) = \frac{1}{z(1-z)}$ valid in the region (i) $|z+1| < 1$

(ii) $1 < |z+1| < 2$(iii) $|z+1| > 2$

Ans: (i)$\sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}} - 1\right)(z+1)^n$ (ii) $\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^{n+1} + \sum_{n=0}^{\infty}\frac{(z+1)^n}{2^{n+1}}$

(iii) $\sum_{n=0}^{\infty}\frac{1-2^n}{(z+1)^{n+1}}$

(6) Find the Laurent's series expansion$f(z) = \frac{1}{z^2+4z+3}$ valid in the region (i) $|z| < 1$and

(ii) $0 < |z+1| < 2$ **Ans:** (i) $f(z) = \frac{1}{2(z+1)} - \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z+1}{2}\right)^n$

(ii) $f(z) = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n [z]^n - \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\left[\frac{z}{3}\right]^n$

(7) Expand $f(z) = \frac{z+3}{(z)(z^2-z-2)}$ the Laurent's series for the region

(i) $|z| < 1$(ii) $1 < |z| < 2$(iii) $|z| < 2$

Ans: (i) $f(z) = \frac{-3}{2z} + \frac{2}{3}\sum_{n=0}^{\infty}(-1)^n(z)^n - \frac{5}{12}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n$

(ii) $f(z) = \frac{-3}{2z} + \frac{2}{3z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n - \frac{5}{12}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$

(iii)$f(z) = \frac{-3}{2z} + \frac{2}{3z}\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n\right] - \frac{5}{6z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$

(8) Expand $\frac{z^2+6z-1}{(z-1)(z-3)(z+2)}$ in $3 < |z+2| < 5$as a Laurent's Series.

Ans: $f(z) = \frac{-1}{z+1} + \frac{8}{5}\sum_{n=0}^{\infty}\left(\frac{3}{z+2}\right)^n - \frac{13}{25}\sum_{n=0}^{\infty}\left(\frac{z+2}{5}\right)^n$

(9) Obtain the Laurent's Series expansion of the function $\frac{e^z}{(z-1)^2}$ in the neighbourhood of its singular point.

Hence find the residue at that pole. **Ans:** The residue is 'e'.

(10) Find the residues of $f(z) = \frac{z^2}{(z-1)^2(z-2)}$ at its isolated singularities using Laurent's Series expansions.

Ans: Residue at $z = -1$ is – 3, Residue at $z = 2$ is 4