1.4 REDUCTION OF QUADRATIC FORM TO CANON ICAL FORM BY ORTHOGONAL TRANSFORMATION

Quadratic Form

A homogeneous polynomial of second degree in any number of variables is called a quadratic form.

The general Quadratic form in three variables $\{x_1, x_2, x_3\}$ is given by

$$f(x_1, x_2, x_3) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{21}x_1x_2 + a_{22}x_2^2 + a_{23}x_2x_3 + a_{31}x_3x_1 + a_{32}x_2x_2 + a_{33}x_3^2$$

This Quadratic form can written as $f(x_1, x_2, x_3) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}x_iy_j$

$$f(x_1, x_2, x_3) = (x_1 \quad x_2 \quad x_3)\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= X'AX$$

Where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and A is called the matrix of the Quadratic form.

Note: To write the matrix of a quadratic form as

$$A = \begin{pmatrix} \text{coeff. of}x^2 & 1/2\text{coeff. of}xy & 1/2\text{coeff. of}xz \\ 1/2\text{coeff. of}xy & \text{coeff. of}y^2 & 1/2\text{coeff. of}yz \\ 1/2\text{coeff. of } xz & 1/2\text{coeff. of}yz & \text{coeff. of } z^2 \end{pmatrix}$$

Example: Write down the Quadratic form in to matrix form

(i)$2x^2 + 3y^2 + 6xy$

Solution:

$$A = \begin{pmatrix} \text{coeff. of } x^2 & 1/2\text{coeff. of } xy \\ 1/2\text{coeff. of } xy & \text{coeff. of}y^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ 3 & 3 \end{pmatrix}$$

(ii) $2x^2 + 5y^2 - 6z^2 - 2xy - yz + 8zx$

Solution:

$$A = \begin{pmatrix} \text{coeff. of} x^2 & 1/2\text{coeff. of} xy & 1/2\text{coeff. of} xz \\ 1/2\text{coeff. of} xy & \text{coeff. of} y^2 & 1/2\text{coeff. of} yz \\ 1/2\text{coeff. of } xz & 1/2\text{coeff. of} yz & \text{coeff. of } z^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & 4 \\ -1 & 5 & -1/2 \\ 4 & -1/2 & -6 \end{pmatrix}$$

Example: Write down the matrix form in to Quadratic form

(i) $\begin{pmatrix} \mathbf{2} & \mathbf{1} & \mathbf{-3} \\ \mathbf{1} & \mathbf{-2} & \mathbf{3} \\ \mathbf{-3} & \mathbf{-2} & \mathbf{5} \end{pmatrix}$

Solution:

Quadratic form is $2x_1^2 - 2x_2^2 + 6x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$

(ii) $\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{3} & \mathbf{1} \\ \mathbf{2} & \mathbf{1} & \mathbf{6} \end{pmatrix}$

Solution:

Quadratic form is $x_1^2 + 3x_2^2 + 6x_3^2 + 2x_1x_2 + 4x_1x_3 + 2x_2x_3$.

Example: Reduce the Quadratic form $\mathbf{x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3 + 6x_2x_3}$ **to canonical form through an orthogonal transformation .Find the nature rank, index, signature and also find the non zero set of values which makes this Quadratic form as zero.**

Solution:

Given $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 1 + 2 + 1 = 4$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 1 + 1 + 1 = 3$$

$$s_3 = |A| = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

Characteristic equation is $\lambda^3 - 4\lambda^2 + 3\lambda = 0$

$$\Rightarrow \lambda = 0\ ; (\lambda^2 - 4\lambda + 3) = 0$$

$$\Rightarrow \lambda = 0,1,3$$

To find the Eigen vectors:

Case (i) When $\lambda = 0$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 - x_2 + 0x_3 = 0 \ldots (1)$$

$$-x_1 + 2x_2 + x_3 = 0 \ldots (2)$$

$$0x_1 + x_2 + x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{2-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 3$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1-3 & -1 & 0 \\ -1 & 2-3 & 1 \\ 0 & 1 & 1-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - x_2 + 0x_3 = 0 \ldots (4)$$

$$-x_1 - x_2 + x_3 = 0 \ldots (5)$$

$$0x_1 + x_2 - 2x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{-1} = \frac{x_2}{2} = \frac{x_3}{2-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$X_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Case (iii) When $\lambda = 1$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1-1 & -1 & 0 \\ -1 & 2-1 & 1 \\ 0 & 1 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0x_1 - x_2 + 0x_3 = 0 \ldots (7)$$

$$-x_1 + x_2 + x_3 = 0 \ldots (8)$$

$$0x_1 + x_2 + 0x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{-1-0} = \frac{x_2}{0-0} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$; $X_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$; $X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

To check $X_1, X_2 \& X_3$ are orthogonal

$$X_1^T X_2 = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = -1 + 2 - 1 = 0$$

$$X_2^T X_3 = (-1 \quad 2 \quad 1)\begin{pmatrix}1\\0\\1\end{pmatrix} = -1+0+1 = 0$$

$$X_3^T X_1 = (1 \quad 0 \quad 1)\begin{pmatrix}1\\1\\-1\end{pmatrix} = 1+0-1 = 0$$

Normalized Eigen vectors are

$$\begin{pmatrix}\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{-1}{\sqrt{3}}\end{pmatrix}\begin{pmatrix}\frac{-1}{\sqrt{6}}\\ \frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\ 0\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0\\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$N^T = \begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}}\\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}$$

Thus the diagonal matrix $D = N^T A N$

$$= \begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0\\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1 & -1 & 0\\ -1 & 2 & 1\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}}\\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$D = \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 3\end{pmatrix}$$

Canonical form $= Y^T DY$ where $Y = \begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$

$$Y^T DY = (y_1, y_2, y_3)\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 3\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$$

$$= 0{y_1}^2 + {y_2}^2 + 3{y_3}^2$$

Rank = 2

Index = 2

Signature = 2 – 0 = 2

Nature is positive semi definite.

To find non zero set of values:

Consider the transformation X = NY

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$x_1 = \frac{y_1}{\sqrt{3}} - \frac{y_2}{\sqrt{6}} + \frac{y_3}{\sqrt{2}}$$

$$x_2 = \frac{y_1}{\sqrt{3}} + \frac{2y_2}{\sqrt{6}} + 0y_3$$

$$x_3 = \frac{-y_1}{\sqrt{3}} + \frac{y_2}{\sqrt{6}} + \frac{y_3}{\sqrt{2}}$$

Put $y_2 = 0$ & $y_3 = 0$

$$x_1 = \frac{y_1}{\sqrt{3}};\ x_2 = \frac{y_1}{\sqrt{3}};\ x_3 = \frac{-y_1}{\sqrt{3}}$$

Put $y_1 = \sqrt{3}$

$x_1 = 1; x_2 = 1;\ x_3 = -1$ which makes the Quadratic equation zero.

Example: Reduce the Quadratic form $x_1^2 + x_2^2 + x_3^2 - 2x_1x_2$ to canonical form through an orthogonal transformation .Find the nature rank,index,signature and also find the non zero set of values which makes this Quadratic form as zero

Solution:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$$= 1 + 1 + 1 = 3$$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 1 + 1 + 0 = 2$$

$$s_3 = |A| = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

Characteristic equation is $\lambda^3 - 3\lambda^2 + 2\lambda = 0$

$$\Rightarrow \lambda = 0\,; (\lambda^2 - 3\lambda + 2) = 0$$

$$\Rightarrow \lambda = 0,1,2$$

To find the Eigen vectors:

Case (i) When $\lambda = 0$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 - x_2 + 0x_3 = 0 \ldots (1)$$

$$-x_1 + x_2 + 0x_3 = 0 \ldots (2)$$

$$0x_1 + 0x_2 + x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{1-0} = \frac{x_2}{0+1} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{0}$$

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Case (ii) When $\lambda = 1$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1-1 & -1 & 0 \\ -1 & 1-1 & 0 \\ 0 & 0 & 1-0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0x_1 - x_2 + 0x_3 = 0 \ldots (4)$$

$$-x_1 + 0x_2 + 0x_3 = 0 \ldots (5)$$

$$0x_1 + 0x_2 + 0x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$X_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

Case (iii) When $\lambda = 2$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1-2 & -1 & 0 \\ -1 & 1-2 & 0 \\ 0 & 0 & 1-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x_1 - x_2 + 0x_3 = 0 \ldots (7)$$

$$-x_1 - x_2 + 0x_3 = 0 \ldots (8)$$

$$0x_1 + 0x_2 - x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{1-0} = \frac{x_2}{0-1} = \frac{x_3}{0-0}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

$$X_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$; $X_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$; $X_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

To check $X_1, X_2 \& X_3$ are orthogonal

$$X_1^T X_2 = (1 \quad 1 \quad 0) \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = 0 + 0 + 0 = 0$$

$$X_2^T X_3 = (0 \quad 0 \quad -1)\begin{pmatrix}1\\-1\\0\end{pmatrix} = 0+0+0=0$$

$$X_3^T X_1 = (1 \quad -1 \quad 0)\begin{pmatrix}1\\1\\0\end{pmatrix} = 1-1+0=0$$

Normalized Eigen vectors are

$$\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{pmatrix}\begin{pmatrix}0\\0\\-1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}\\0\end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}}\\0 & -1 & 0\end{pmatrix}$$

$$N^T = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\0 & 0 & -1\\\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{pmatrix}$$

Thus the diagonal matrix $D = N^T A N$

$$= \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\0 & 0 & -1\\\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{pmatrix}\begin{pmatrix}1 & -1 & 0\\-1 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}}\\0 & -1 & 0\end{pmatrix}$$

$$D = \begin{pmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 2\end{pmatrix}$$

Canonical form $= Y^T DY$ where $Y = \begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$

$$Y^T DY = (y_1, y_2, y_3)\begin{pmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 2\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$$

$$= 0{y_1}^2 + {y_2}^2 + 2{y_3}^2$$

Rank = 2

Index = 2

Signature = 2 - 0 = 2

Nature is positive semi definite.

To find non zero set of values:

Consider the transformation X = NY

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$x_1 = \frac{y_1}{\sqrt{2}} + 0 + \frac{y_3}{\sqrt{2}}$$

$$x_2 = \frac{y_1}{\sqrt{2}} + 0 - \frac{y_3}{\sqrt{2}}$$

$$x_3 = 0 - y_2 - 0$$

Put $y_2 = 0$ & $y_3 = 0$

$x_1 = \frac{y_1}{\sqrt{2}}$; $x_2 = \frac{y_1}{\sqrt{2}}$; $x_3 = 0$

Put $y_1 = \sqrt{2}$

$x_1 = 1; x_2 = 1; x_3 = 0$ which makes the Quadratic equation zero.

Example: Reduce the Quadratic form $2x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 - 2x_1x_3 - 4x_2x_3$ to canonical form through an orthogonal transformation .

Solution:

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

s_1 = sum of the main diagonal element

$= 2 + 1 + 1 = 4$

s_2 = sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= -3 + 1 + 1 = -1$$

$$s_3 = |A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{vmatrix} = -4$$

Characteristic equation is $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

$$\lambda = -1,1,4$$

To find the Eigen vectors:

Case (i) When $\lambda = -1$ the Eigen vector is given by $(A - \lambda I)X = 0$

$$\text{where } X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2+1 & 1 & -1 \\ 1 & 1+1 & -2 \\ -1 & -2 & 1+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3x_1 + x_2 - x_3 = 0 \ldots (1)$$

$$x_1 + 2x_2 - 2x_3 = 0 \ldots (2)$$

$$-x_1 - 2x_2 + 2x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{-2+2} = \frac{x_2}{-1+6} = \frac{x_3}{6-1}$$

$$\frac{x_1}{0} = \frac{x_2}{5} = \frac{x_3}{5}$$

$$X_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 1$ the Eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-1 & 1 & -1 \\ 1 & 1-1 & -2 \\ -1 & -2 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + x_2 - x_3 = 0 \ldots (4)$$

$$x_1 + 0x_2 - 2x_3 = 0 \ldots (5)$$

$$-x_1 - 2x_2 + 0x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{-2+0} = \frac{x_2}{-1+2} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Case (iii) When $\lambda = 4$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-4 & 1 & -1 \\ 1 & 1-4 & -2 \\ -1 & -2 & 1-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + x_2 - x_3 = 0 \ldots (7)$$

$$x_1 - 3x_2 - 2x_3 = 0 \ldots (8)$$

$$-x_1 - 2x_2 - 3x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{-2-3} = \frac{x_2}{-1-4} = \frac{x_3}{6-1}$$

$$\frac{x_1}{-5} = \frac{x_2}{-5} = \frac{x_3}{5}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; $X_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$;

$$X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

To check X_1, X_2& X_3 are orthogonal

$$X_1{}^T X_2 = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 0 - 1 + 1 = 0$$

$$X_2{}^T X_3 = \begin{pmatrix} 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 2 - 1 - 1 = 0$$

$$X_3{}^T X_1 = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 + 1 - 1 = 0$$

Normalized Eigen vectors are

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$$

Thus the diagonal matrix $D = N^T AN$

$$= \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Canonical form $= Y^T DY$ where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$Y^T \mathrm{DY} = (\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} \mathrm{y}_1 \\ \mathrm{y}_2 \\ \mathrm{y}_3 \end{pmatrix}$$

$$= -{y_1}^2 + {y_2}^2 + 4{y_3}^2$$