LTI SYSTEM ANALYSIS USING Z-TRANSFOR M

The Z-Transform of impulse response is called transfer or system function H(Z).

$$Y(Z)=X(Z)H(Z)$$

General form of LCCDE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

ComputingtheZ-Transform

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Example1:Considerthe system described by the difference equation.

$$y[n] = x[n] + \frac{1}{3}x[n-1] + \frac{5}{4}y[n-1] - \frac{1}{2}y[n-2] + \frac{1}{16}y[n-3]$$

Solution:

$$y[n] = x[n] + \frac{1}{3}x[n-1] + \frac{5}{4}y[n-1] - \frac{1}{2}y[n-2] + \frac{1}{16}y[n-3]$$

Here $N=3,\ M=1$. Order 3 homogeneous equation:

$$y[n]-\frac{5}{4}y[n-1]+\frac{1}{2}y[n-2]-\frac{1}{16}y[n-3]=0 \qquad n\geq 2$$

The characteristic equation:

$$1-\frac{5}{4}a^{-1}+\frac{1}{2}a^{-2}-\frac{1}{16}a^{-3}=0$$

The roots of this third order polynomial is: $a_1=a_2=1/2 \quad a_3=1/4$ and

$$y_h[n]=h[n]=A_1(\frac{1}{2})^n+A_2n(\frac{1}{2})^n+A_3(\frac{1}{4})^n, \quad n\geq 2$$

Let us assume $y[-1]=0$ then (3.52) for this case becomes:

$$\begin{bmatrix} a_0 & 0 \\ a_1 & a_0 \end{bmatrix}.\begin{bmatrix} y[0] \\ y[1] \end{bmatrix}=\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 \\ -5/4 & 1 \end{bmatrix}.\begin{bmatrix} y[0] \\ y[1] \end{bmatrix}=\begin{bmatrix} 1 \\ 1/3 \end{bmatrix} \quad \Rightarrow y[0]=1;\ y[1]=19/12$$

with these we have the impulse response of this system:

$$h[n]=-\frac{4}{3}(\frac{1}{2})^n+\frac{10}{3}n(\frac{1}{2})^n+\frac{7}{3}(\frac{1}{4})^n, \quad n\geq 0$$

APPLICATION OF Z TRANSFORM TO DISCRETE SYSTEM -STABILITY ANALYSIS, FREQUENCY RESPONSE 5.4.1.STABILITY ANALYSIS Location of poles for stability: Let h (n) be the impulse response of an LTI discrete time system.Now, if h (n) satisfies

the condition

$$\sum_{n=-\infty}^{\infty}|h(n)| < \infty$$

Then the LTI discrete time system is stable.

The stability condition of above equation can be transformed as a condition on location of poles of transfer function of the LTI discrete time system in z plane.

Let, h (n) = an u(n)

Now, $\sum_{n=-\infty}^{\infty}|h(n)|=\sum_{n=-\infty}^{\infty}|a^n u(n)| = \sum_{n=0}^{\infty} a^n$

If $|a|$ is such that, $0<|a|<1$, then $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ = constant, and so the system is stable.

If $|a|>1$, then $\sum_{n=0}^{\infty} a^n = \infty$ and so the system is unstable.

Now (z) =z {h (n)} =z{$a^n u(n)$}= $\frac{z}{z-a}$

Here H (z) has pole at z=a

If $|a|<1$ then the pole will lie inside the unit circle and if $|a|>1$, then the pole will lie outside the circle.Therefore we can say that, for a stable discrete time system the poles should lie inside the unit circle.

Roc Of A Stable System:

Let H (z) be Z transform of h (n).Now, by the definition of Z transform we get

H (z) = $\sum_{n=-\infty}^{\infty} h(n) z^{-n}$

Let us evaluate H (z) for z=1

H (z) = $\sum_{n=-\infty}^{\infty} h(n)$

On taking absolute value on both sides we get,

$|H(z)|=|\sum_{n=-\infty}^{\infty} h(n)|$ => $|H(z)| = \sum_{n=-\infty}^{\infty}|h(n)|$

For a stable LTI discrete time system,

$\sum_{n=-\infty}^{\infty} h(n)<\infty$ => $|H(z)|<\infty$

Therefore we conclude that z=1 will be a point inside the ROC of a stable system. Hence for a stable discrete time system the ROC of impulse response should include the unit circle.

General condition for stability in z plane

On combining the condition for location of poles and the ROC we can say that for a stable LTI discrete time system the poles should lie inside the unit circle and the unit circle should be included in the ROC of impulse response of the system.

5.4.2. FREQUENCY RESPONS E

In general the input output relation of an LTI discrete time system is represented by the constant coefficient difference equation shown below.

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

The solution of the above difference equation is the total response y (n) of LTI discrete time system, which consists of two parts. In signals and systems the two parts of the solution y(n) are called zero input response $y_{zi}(n)$ and zero-state response $y_{zs}(n)$.

Response, $y(n) = y_{zi}(n) + y_{zs}(n)$.

Zero-input response (or Free response or Natural response) using Z transform:

The Zero-input response $y_{zi}(n)$ is mainly due to initial output in the system. The Zero-input response is obtained from the system equation when input x (n) =0.

On substituting x (n) =0 and y (n) = $y_{zi}(n)$ in equation 1 we get

$$\sum_{m=0}^{N} a_m\, y_{zi}(n-m) = 0; \text{ with } a_0 = 1$$

On taking Z transform of the above equation with non zero initial conditions for output we can form an equation for $Y_{zi}(z)$. The zero-input response $y_{zi}(n)$ of a discrete time system is given by inverse transform of $Y_{zi}(z)$.

Zero-State Response (Or Forced Response)

The zero state response $y_{zs}(n)$ is the response of the system due to input signal and with zero initial output. The Zero-state response is obtained from the difference equation governing the system for specific input signal x (n) for n>=0 with zero initial output.

On substituting y (n) = $y_{zs}(n)$ we get

$$\sum_{m=0}^{N} a_m\, y_{zs}(n-m) = 0 = \sum_{m=0}^{M} b_m\, x(n-m) = 0; \text{ with } a_0 = 1$$

On taking Z transform of the above equation with zero initial conditions for output and nonzero initial conditions for input we can form an equation for Yzs (z).The zero state response yzs (n) of a discrete time system is given by the inverse z transform of Yzs (z).

Total response:

The total response y(n) is the response of the system due input signal and initial output.

Example-1:

Determine the response of discrete time LTI system governed by the difference equation (n) = -0.8y (n-1) +x (n), when the input is unit step and initial condition,

a) y(-1)=0 and b)y(-1)=2/9

Solution:

Given that x (n) = u (n);

X (z) =Z{x (n)} = Z {u (n)} =z/z-1

Given that, y(n) =-0.8y(n-1)+x(n),

y (n)+ 0.8y(n-1)= x(n),

On taking the z transform of the above equation we get,

Y (z) +0.8[Z-1 Y (z) + y (-1)] =X (z)

Y(z)[1+0. 8 Z-1]+0.8 y (-1) = $\frac{z}{z-1}$

Y (z) $(1+\frac{0.8}{z}) = \frac{z}{z-1} - 0.8$ y(-1)

Y (z)$(\frac{z+0.8}{z}) = \frac{z}{z-1} - 0.8$ y(-1)

Y(z) = $\frac{z2}{(z-1)(z+0.8)}$ - 0.8 $\frac{zy(-1)}{(z+0.8)}$

Let P(z) =$\frac{z2}{(z-1)(z+0.8)}$ => $\frac{P(z)}{z} = \frac{z}{(z-1)(z+0.8)}$

Let $\frac{z}{(z-1)(z+0.8)} = \frac{A}{z-1} + \frac{B}{z+0.8}$

A = $\frac{z}{(z-1)(z+0.8)}(z-1)$ | z=1

= $\frac{1}{1+0.8}$

= $\frac{1}{1.8}$

= $\frac{10}{18}$

=5/9

B =

$$\frac{z}{(z-1)(z+0.8)}(z+0.8)\mid z=-0.8$$

$$\frac{0.8}{-0.8-1}$$

$$=\frac{0.8}{1.8}$$

$$\frac{8}{18}$$

$$=\frac{4}{9}$$

$$\frac{P(z)}{z}=\frac{z}{(z-1)(z+0.8)}$$

$$=\frac{5}{9}\frac{1}{z-1}+\frac{4}{9}\frac{1}{z+0.8}$$

$$Y(z)=\frac{5}{9}\frac{z}{z-1}+\frac{4}{9}\frac{z}{z+0.8}-0.8\frac{zy(-1)}{(z+0.8)}$$

a) When y(-1)=0

$$Y(z)=\frac{5}{9}\frac{z}{z-1}+\frac{4}{9}\frac{z}{z+0.8}$$

Response (n)=$z^{-1}\{Y(z)\} = z^{-1}\{\frac{5}{9}\frac{z}{z-1}+\frac{4}{9}\frac{z}{z+0.8}\}$

$$=\frac{5}{9}u(n)+\frac{4}{9}(-0.8)^n u(n)$$

b) When y (-1) = 2/9

When y(-1) = 2/9,we get

$$Y(z)=\frac{5}{9}\frac{z}{z-1}+\frac{4}{9}\frac{z}{z+0.8}-0.8\frac{zy(-1)}{(z+0.8)}$$

$$=\frac{5}{9}\frac{z}{z-1}+\frac{4}{9}\frac{z}{z+0.8}-0.8\times\frac{2}{9}\frac{z}{z+0.8}$$

$$=\frac{5}{9}\frac{z}{z-1}+\frac{2.4}{9}\frac{z}{z+0.8}$$

$$=\frac{5}{9}\frac{z}{z-1}\quad\frac{24}{90}\frac{z}{z+0.8}$$

$$=\frac{5}{9}\frac{z}{z-1}+\frac{12}{45}\frac{z}{z+0.8}$$

Response (n) = z-1 {Y(z)} = z-1$\frac{5}{9}\{\frac{z}{z-1}+\frac{12}{45}\frac{z}{z+0.8}\}$

$$=[\frac{5}{9}+\frac{12}{45}(-0.8)^n]\,u(n)$$

$$\text{Response}=[\frac{5}{9}+\frac{12}{45}(-0.8)^n]\,u(n)$$