Boundary Conditions Between Two Perfect Dielectrics

Consider two boundary between two perfect dielectrics. One has permittivity ε_1 while other has permittivity ε_2.

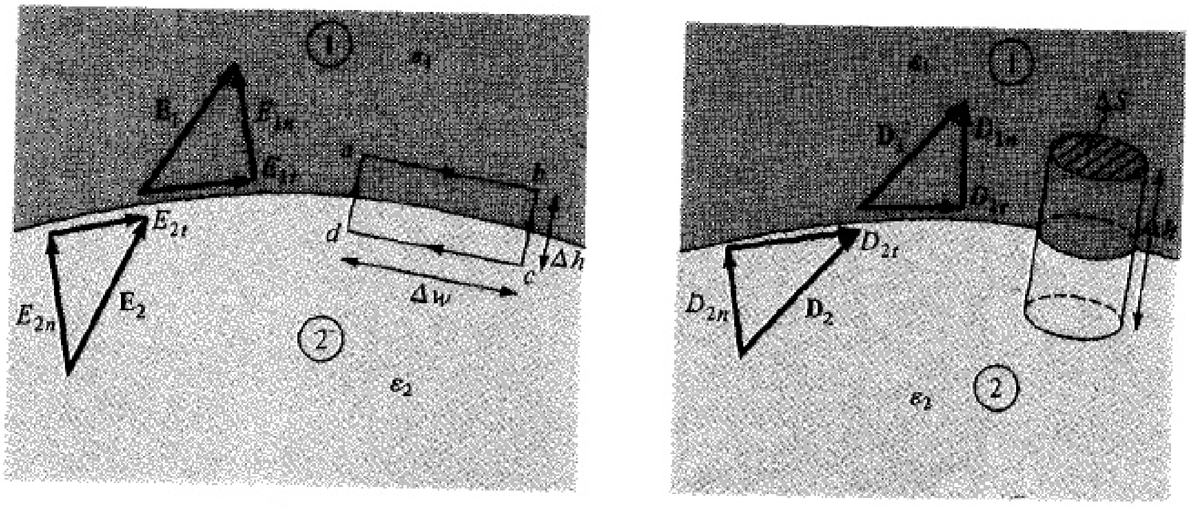
The $\bar{E}$ and $\bar{D}$ have two components

*Tangential to the boundary.

*Normal to the surface.

Consider a closed path abcda rectangular in shape having height Δh and width Δw.

$\oint E.dl = 0$ ---------1



(a) (b)

Fig 17 : Dielectric – dielectric boundary

$\int_a^b E.dl + \int_b^c E.dl + \int_c^d E.dl + \int_d^a E.dl = 0$ ----------2

The total Electric field in region 1 is

$\overline{E_1} = \overline{E_{1t}} + \overline{E_{1N}}$

Total field in region 2 is

$\overline{E_2} = \overline{E_{2t}} + \overline{E_{2N}}$

Let $|E_{1t}| = E_{tan1}$, $|E_{2t}| = E_{tan2}$

$|E_{1N}| = E_{1N}$, $|E_{2N}| = E_{2N}$

To analyze the boundary conditions the rectangle to be reduced at $\Delta h \rightarrow 0$

$$\int_b^c E.dl = 0 = \int_d^a E.dl$$

Eqn 2 is reduced to

$$\int_a^b E.dl + \int_c^d E.dl = 0 \text{ ------- } 3$$

$$\int_a^b E.dl = \int_a^b E_{tan1}\ dl = E_{tan1}\ \Delta w \text{ -------- } 4$$

$$\int_c^d E.dl = \int_c^d E_{tan2}.dl = -E_{tan2}\ \Delta w \text{ -------- } 5$$

Substitute 4 & 5 in equation 3

$E_{tan1}\ \Delta w - E_{tan\ 2}\ \Delta w = 0$

$\Delta w\ [E_{tan1} - E_{tan2}] = 0$

$E_{tan1} - E_{tan2} = 0$

$E_{tan1} = E_{tan2} = 0$ -------- 6

Thus the tangential components of field intensity $\bar{E}$ at the boundary in both the dielectrics remain same. Ie electric field intensity is continuous across the boundary

$D = \varepsilon E$

$D_{tan1} = \varepsilon_1 E_{tan\ 1}$ & $D_{tan2} = \varepsilon_2 E_{tan\ 2}$ ----------- 7

$$\frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1 E_{tan1}}{\varepsilon_2 E_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}} \text{ --------8}$$

Thus the tangential components of D undergoes some change across the interface hence tangential D is said to be discontinuous across the boundary.

Normal components:

Use Gauss 's law to find normal components.

$$\oint D.ds = Q_{enc} \text{ -------------9}$$

$$\int_{top} D.ds + \int_{bottom} D.ds + \int_{lateral} D.ds = Q_{enc} \text{ --------------10}$$

$\int_{lateral} D.ds$ =0 as $\Delta h \rightarrow 0$ -------A

$\int_{top} D.ds + \int_{bottom} D.ds$ =Q

$\int_{top} D.ds$ = $D_{N1}\,\Delta S$ ---------11

For top surface the direction of D_N is entering the boundary while for bottom surface the direction D_N is leaving the boundary. Both are opposite in direction at the boundary.

$\int_{bottom} D.ds$ = $-D_{N2}\,\Delta s$ ------12

$Q_{en} = \rho_s\, \Delta s$ -------13

Sub 11, 12 & 13 & A in Equation 10

$D_{N1}\Delta s$ - $D_{N2}\,\Delta s$= $\rho_s \Delta s$

Δs $(D_{N1}$-$D_{N2})$ = ρ_s ------14

There is no free charge available in perfect dielectric .All the charges in dielectric are bound charges and are not free. Hence ρ_s = 0.

Equation 14 becomes

D_{N1} - D_{N2} = 0 --------- 15

$$D_{N1} = D_{N2}$$

Hence the normal component of flux density is continuous at the boundary between two perfect dielectrics.

D_{N1}= ε_1 E_{N1} & D_{N2}= ε_2 E_{N2}

$$\frac{\mathrm{D_{N1}}}{\mathrm{D_{N2}}} = \frac{\varepsilon_1 E_{N1}}{\varepsilon_2 E_{N2}} = 1$$

$$\frac{E_{N1}}{E_{N2}} = \frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}}$$

The normal components of E are inversely proportional to the relative permittivity's of the two media

Refraction of D at the boundary:

The directions of $\bar{D}$ and $\bar{E}$ change at the boundary between two dielectrics

Let $\overline{D_1}$ and $\overline{E_1}$ make an angle θ_1 normal to the surface.

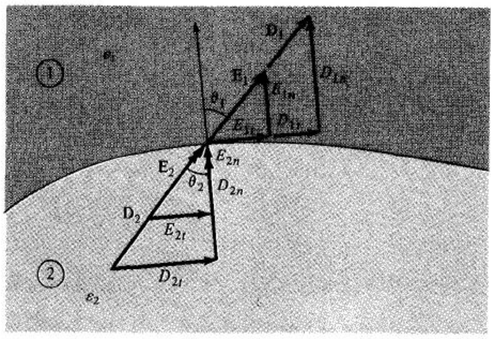


Fig 18: Refraction of D or E at a dielectric – dielectric boundary

$|\overline{D_1}| = D_1$ & $|\overline{D_2}| = D_2$

$D_{N1} = D_1 \cos\theta_1$ ------- 16

$D_{N2} = D_2 \cos\theta_2$ --------17

But

$D_{N1} = D_{N2}$

$D_1 \cos\theta_1 = D_2 \cos\theta_2$ ------18

While $D_{tan\,1} / D_{tan\,2} = \varepsilon_1 / \varepsilon_2$

But $D_{tan\,1} = D_1 \sin\theta_1$ & $D_{tan\,2} = D_2 \sin\theta_2$ --------- 19

$D_1 \sin\theta_1 / D_2 \sin\theta_2 = \varepsilon_1 / \varepsilon_2 = D_{tan\,1} / D_{tan\,2}$

$$\frac{D_{tan1}}{D_{N1}} = \frac{D_1 Sin\ \theta_1}{D_1 Cos\ \ \theta_1} = \tan\theta_1$$

$$\frac{D_{tan2}}{D_{N2}} = \frac{D_2 Sin\ \theta_2}{D_2 Cos\ \ \theta_2} = \tan\theta_2$$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{D_{tan1}}{D_{N1}}\ X\ \frac{D_{N2}}{D_{tan2}}$$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{D_{tan1}}{D_{tan2}}\ X\ \frac{D_{N2}}{D_{N1}} \qquad \text{---------------- } 20$$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2}$$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}} \qquad \text{--------------- 21}$$

This is called law of refraction

If $\varepsilon_1 > \varepsilon_2$, then $\theta_1 > \theta_2$.

The magnitude of $\overline{D}$ in region 2 can be obtained as

$D_2^2 = D_{N2}^2 + D_{tan\,2}^2$

$== (D_1 \cos\theta_1)^2 + D_{tan\,2}^2$ ------ 22

$$D_{tan2} = D_{tan1}\, X \frac{\varepsilon_2}{\varepsilon_1} = D_1\, Sin\,\theta_1\, \frac{\varepsilon_2}{\varepsilon_1}$$

$$D_2^2 = (D_1 Cos\,\theta_1)^2 + \left(D_1\, Sin\,\theta_1\, \frac{\varepsilon_2}{\varepsilon_1}\right)^2$$

$$D_2 = D_1\sqrt{Cos^2\theta_1 + Sin^2\theta_1\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2} \qquad \text{------- 23}$$

Similarly E_2 is

$$E_2 = E_1\sqrt{Sin^2\theta_1 + Cos^2\theta_1\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2} \qquad \text{------ 24}$$

1. D is larger in the region of larger permittivity.
2. E is larger in the region of smaller permittivity.
3. $|\overline{D_1}| = |\overline{D_2}|$ if $\theta_1 = \theta_2 = 0^o$
4. $|\overline{E_1}| = |\overline{E_2}|$ if $\theta_1 = \theta_2 = 90^o$

Poisson’s and Laplace’s Equation

Poisson’s equation can be derived from point of Gauss’s law.

$$\nabla . D = \rho_v \qquad \text{----------}①$$

$\overline{D}$ → Electric flux density, ρ_v → Volume charge density

For a homogeneous, linear and isotropic medium flux density and electric field intensity are directly proportional

$$\overline{D} = \varepsilon \overline{E} \qquad \text{-----------}②$$

Substitute in ① $\Delta . \varepsilon E = \rho_v$ ---------③

From gradient relationship

$E = -\nabla \mathrm{v}$ -----=----④

Substitute ④ in ③

③=> $\nabla . \varepsilon(-\nabla v) = \rho_v$

$-\varepsilon \rightarrow$ constant taking outside

$$-\varepsilon(\nabla . \nabla v) = \rho_v$$

$$\nabla . \nabla v = -\frac{\rho_v}{\varepsilon}$$ ----------⑤

$$\nabla^2 v = -\frac{\rho_v}{\varepsilon}$$

This equation is called poissons equation.

In certain region (dielectric) the volume charge density is zero.

$\nabla^2 v = 0$ (For charge free region)

This is special case of Poisson's equation and is called Laplace equation .The ∇^2 operation is called the Laplacian of ϑ

Laplace equation in Cartesian co-ordinate

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$

Cylindrical co-ordinate:

$$\nabla^2 v = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial v}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 v}{\partial \emptyset^2}\right) + \frac{\partial^2 v}{\partial z^2} = 0$$

Spherical co-ordinate:

$$\nabla^2 v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{r^2 sin\,\theta}\frac{\partial}{\partial \theta}\left(sin\,\theta\,\frac{\partial v}{\partial \theta}\right) + \frac{1}{r^2 sin^2\theta}\left(\frac{\partial^2 v}{\partial \emptyset^2}\right) = 0$$