

Discrete Time Fourier Transform and Its Properties

The DTFT is a transformation that maps Discrete-time (DT) signal $x[n]$ into a complex valued function of the real variable namely:

$$F[x(n)] = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

INVERSE DISCRETE FOURIER TRANSFORM

IDTFT is given by

$$x(n) = F^{-1}[X(e^{j\omega})] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega, \text{ for } n=-\infty \text{ to } \infty$$

Example 1: Find the DTFT of $x(n) = a^n u(n)$.

Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}.$$

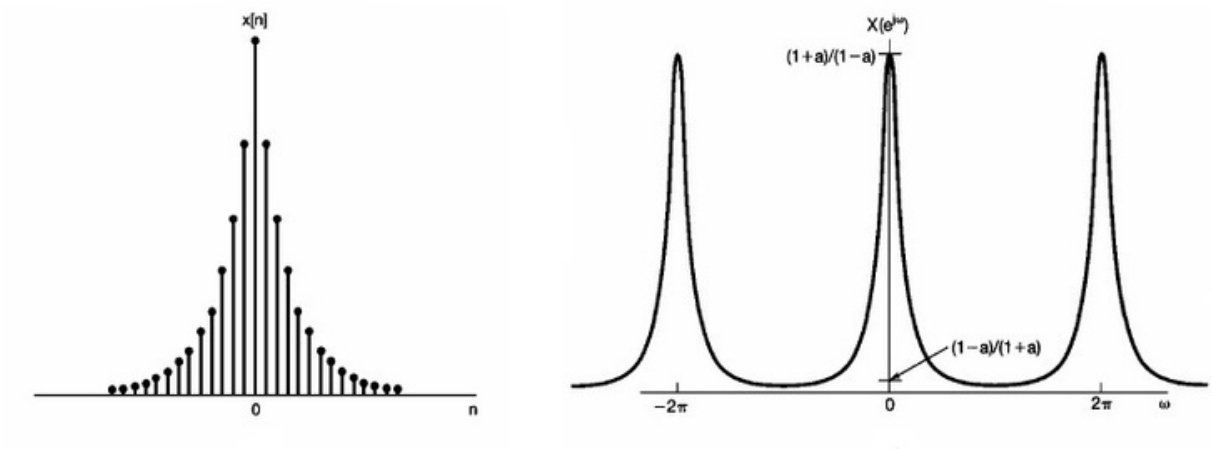
Example 2: Find the DTFT of $x(n) = a^{|n|}$, $|a| < 1$.

Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} u[n] e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

Let $m = -n$ in the first summation, we obtain

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} a^{|n|} u[n] e^{-j\omega n} = \sum_{m=1}^{\infty} a^m e^{j\omega m} + \sum_{n=0}^{\infty} a^n e^{-j\omega n} \\ &= \frac{ae^{j\omega}}{1 - ae^{j\omega}} + \frac{1}{1 - ae^{-j\omega}} = \frac{1 - a^2}{1 - 2a \cos \omega + a^2} \end{aligned}$$



PROPERTIES OF DTFT:

Linearity:

$$\mathcal{F}_{DT}\{x_1(n) + x_2(n)\} = \mathcal{F}_{DT}\{x_1(n)\} + \mathcal{F}_{DT}\{x_2(n)\} = X_1(e^{j\omega}) + X_2(e^{j\omega})$$

Time Shifting:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\{x(n - n_0)\} = e^{-j\omega n_0} X(e^{j\omega})$$

Proof:

$$\sum_{n=-\infty}^{\infty} x(n - n_0) e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x(m) e^{-j\omega(m+n_0)} = e^{-j\omega n_0} \sum_{m=-\infty}^{\infty} x(m) e^{-j\omega m}$$

Time Reversal:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\{x(-n)\} = X(e^{-j\omega})$$

Convolution:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$ and $\mathcal{F}_{DT}\{y(n)\} = Y(e^{j\omega})$, then:

$$G(e^{j\omega}) = \mathcal{F}_{DT}\{x(n) * y(n)\} = X(e^{j\omega}) Y(e^{j\omega})$$

Proof:

$$\begin{aligned} G(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(m) y(n-m) e^{-jm\omega} = \sum_{n=-\infty}^{\infty} x(m) \sum_{m=-\infty}^{\infty} y(n-m) e^{-jm\omega} \\ &= \sum_{m=-\infty}^{\infty} x(m) \sum_{r=-\infty}^{\infty} y(r) e^{-j(m+r)\omega} = \sum_{m=-\infty}^{\infty} x(m) e^{-jm\omega} \sum_{r=-\infty}^{\infty} y(r) e^{-jr\omega} \end{aligned}$$

Frequency Shifting:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\left\{e^{j\omega_0 n} x(n)\right\} = X\left(e^{j(\omega-\omega_0)}\right),$$

Time Multiplication:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\left\{nx(n)\right\} = -z \frac{dX(z)}{dz} \Bigg|_{z=e^{j\omega}}$$

Parseval's Theorem:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$ and $\mathcal{F}_{DT}\{y(n)\} = Y(e^{j\omega})$, then:

$$\sum_{n=-\infty}^{\infty} x(n) y^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) Y^*(e^{j\omega}) d\omega$$

Proof:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\sum_{n=-\infty}^{\infty} x(n) e^{-jn\omega} \right] Y^*(e^{j\omega}) d\omega = \sum_{n=-\infty}^{\infty} x(n) \frac{1}{2\pi} \int_{-\pi}^{\pi} Y^*(e^{j\omega}) e^{-j\omega n} d\omega$$

For the case $x(n) = y(n)$, then:

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

Multiplication of Sequences:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$ and $\mathcal{F}_{DT}\{y(n)\} = Y(e^{j\omega})$, then:

$$\begin{aligned} \mathcal{F}_{DT}\{x(n)y(n)\} &= \sum_{n=-\infty}^{\infty} x(n)y(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(n) \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j\lambda}) e^{j\lambda n} d\lambda \right] e^{-j\omega n} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j\lambda}) d\lambda \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega-\lambda)n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j\lambda}) X(e^{j(\omega-\lambda)}) d\lambda \\ &= \frac{1}{2\pi} Y(e^{j\omega}) * X(e^{j\omega}) \end{aligned}$$

Differentiation in the Frequency Domain:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\frac{dX(e^{j\omega})}{d\omega} = \frac{d}{d\omega} \left[\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \right] = -j \sum_{n=-\infty}^{\infty} nx(n) e^{-j\omega n} = -j \mathcal{F}_{DT}\{nx(n)\}$$