

1.3 CAYLEY HAMILTON THEOREM

Cayley Hamilton Theorem

Statement: Every square matrix satisfies its own characteristic equation.

Uses of Cayley Hamilton Theorem:

To calculate (i) the positive integral power of A and

(ii) the inverse of a non-singular square matrix A.

Example: Show that the matrix $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ satisfies its own characteristic equation.

Solution:

$$\text{Let } A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

The characteristic equation of the given matrix is $|A - \lambda I| = 0$

$$\lambda^2 - S_1\lambda + S_2 = 0$$

Where $S_1 =$ sum of the main diagonal elements.

$$= 1 + 1 = 2$$

$$S_2 = |A| = \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 1 + 4 = 5$$

\therefore The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$

To prove: $A^2 - 2A + 5I = 0$

$$A^2 = AA = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix}$$

$$A^2 - 2A + 5I = \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix} - 2 \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} + 5 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix} + \begin{pmatrix} -2 & 4 \\ -4 & -2 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

Therefore, the given matrix satisfies its own characteristic equation.

Example: Verify Cayley – Hamilton theorem find A^4 and A^{-1} when $A =$

$$\begin{bmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Solution:

The characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\lambda^3 - S_1\lambda^2 + S_2\lambda = 0$ where

$$S_1 = \text{sum of its leading diagonal elements} = 2 + 2 + 2 = 6$$

$S_2 = \text{sum of the minors of its leading diagonal elements}$

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (4 - 1) + (4 - 2) + (4 - 1) = 3 + 2 + 3 = 8$$

$$S_3 = |A| = 2(4 - 1) + 1(-2 + 1) + 2(1 - 2)$$

$$= 2(3) + 1(-1) + 2(-1) = 6 - 1 - 2 = 3$$

\therefore The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$\text{i.e., } \lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$$

By Cayley-Hamilton theorem

[Every square matrix satisfies its own characteristic equation]

$$(i.e.) A^3 - 6A^2 + 8A - 3I = 0 \quad \dots (1)$$

Verification:

$$A^2 = A \times A = \begin{bmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = A \times A^2 = \begin{bmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$$\therefore A^3 - 6A^2 + 8A - 3I = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - 6 \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$+ 8 \begin{bmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix}$$

$$+ \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

To find A^4 :

$$(1) \Rightarrow A^3 - 6A^2 - 8A + 3I \dots (2)$$

Multiply A on both sides, we get

$$A^4 = 6A^3 - 8A^2 + 3A = 6[6A^2 - 8A + 3I] - 8A^2 + 3A \text{ by (2)}$$

$$= 36A^2 - 48A + 18I - 8A^2 + 3A$$

$$A^4 = 28A^2 - 45A + 18I \dots (3)$$

$$(1) \Rightarrow A^4 = 28 \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 45 \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 18 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{bmatrix} - \begin{bmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{bmatrix} +$$

$$\begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

To find A^{-1} :

$$(1) \times A^{-1} \Rightarrow A^2 - 6A + 8I - 3A^{-1} = 0$$

$$3A^{-1} = A^2 - 6A + 8I$$

$$3A^{-1} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6 \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} + \begin{bmatrix} -12 & 6 & -12 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$3A^{-1} = \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{3} \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

Example: Find A^{-1} if $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$, using Cayley- Hamilton theorem.

Solution:

The characteristic equation of A is $|A - \lambda I| = 0$

(i. e.) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 =$ sum of its leading diagonal elements

$$= 1 + 2 + (-1) = 2$$

$S_2 =$ sum of the minors of its leading diagonal elements

$$\begin{aligned} &= \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} \\ &= (-2 + 1) + (-1 - 8) + (2 + 3) \\ &= (-1) + (-9) + 5 = -5 \end{aligned}$$

$$\begin{aligned} S_3 = |A| &= \begin{vmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix} \\ &= 1(-2 + 1) + 1(-3 + 2) + (3 - 4) \\ &= 1(-1) + 1(-1) + 4(-1) \\ &= -1 - 1 - 4 = -6 \end{aligned}$$

\therefore The Characteristic equation is $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

By Cayley Hamilton Theorem we get

[Every square matrix satisfies its own characteristic equation]

$$\therefore A^3 - 2A^2 - 5A + 6I = 0 \quad \dots (1)$$

To find A^{-1}

$$(1) \times A^{-1} \Rightarrow A^2 - 2A - 5I + 6A^{-1} = 0$$

$$A^2 - 2A - 5I + 6A^{-1} = 0$$

$$6A^{-1} = -A^2 + 2A + 5I$$

$$A^{-1} = \frac{1}{6}[-A^2 + 2A + 5I] \quad \dots (2)$$

$$A^2 = A \times A$$

$$= \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3+8 & -1-2+4 & 4+1-4 \\ 3+6-2 & -3+4-1 & 12-2+1 \\ 2+3-2 & -2+2-1 & 8-1+1 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{bmatrix}$$

$$-A^2 + 2A + 5I = \begin{bmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{bmatrix} + 2 \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 8 \\ 6 & 4 & -2 \\ 4 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$$

$$\text{From (2)} \Rightarrow A^{-1} = \frac{1}{6} \begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$$

Example: Use Cayley – Hamilton theorem to find the value of the matrix given by

(i) $f(A) = A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

(ii) $A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$ if the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution:

The characteristic equation of A is $|A - \lambda I| = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \text{ where}$$

$S_1 =$ sum of the main diagonal elements

$$= 2 + 1 + 2 = 5$$

$S_2 =$ sum of the minors of main diagonal elements

$$= \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= (2 - 0) + (4 - 1) + (2 - 0) = 2 + 3 + 2 = 7$$

$$= (-1) + (-9) + 5 = -5$$

$$S_3 = |A| = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 2(2 - 0) - 1(0 - 0) + 1(0 - 1) = 4 - 1 = 3$$

Therefore, the characteristic equation is

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By C - H theorem, we get

$$A^3 - 5A^2 + 7A - 3I = 0 \dots (1)$$

Let

$$i) f(A) = A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$ii) g(A) = A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$$

$$(i) \begin{array}{l} A^5 + A \\ \hline A^3 - 5A^2 + 7A - 3I \left| \begin{array}{l} A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \\ A^8 - 5A^7 + 7A^6 - 3A^5 \\ \hline A^4 - 5A^3 + 8A^2 - 2A \\ A^4 - 5A^3 + 7A^2 - 3A \\ \hline A^2 + A + 1I \end{array} \right. \end{array}$$

$$\begin{aligned} f(A) &= (A^3 - 5A^2 + 7A - 3I)(A^2 + A) + A^2 + A + I \\ &= 0 + A^2 + A + I \text{ by (1)} \\ &= A^2 + A + I \dots (2) \end{aligned}$$

Now,

$$\begin{aligned} A^2 &= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} \end{aligned}$$

$$\therefore A^2 + A + I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

(ii) $A^5 + 8A + 35I$

$$\begin{array}{r|l}
 A^3 - 5A^2 + 7A - 3I & \begin{array}{l} A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + 1I \\ A^8 - 5A^7 + 7A^6 - 3A^5 \\ \hline 8A^4 - 5A^3 + 8A^2 - 2A \\ 8A^4 - 40A^3 + 56A^2 - 24A \\ \hline 35A^3 - A^2 + 22A + 1I \\ 35A^3 - 175A^2 + 245A - 105I \\ \hline 127A^2 - 223A + 106I \end{array} \\
 (-) & \\
 (-) & \\
 (-) &
 \end{array}$$

$$\begin{aligned}
 g(A) &= (A^3 - 5A^2 + 7A - 3I)(A^4 + 8A + 35I) + 127A^2 - 223A + 106I \\
 &= 0 + 127A^2 - 223A + 106I \\
 &= 127A^2 - 223A + 106I
 \end{aligned}$$

$$= 127 \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 223 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 106 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$g(A) = \begin{bmatrix} 295 & 285 & 285 \\ 0 & 10 & 0 \\ 285 & 285 & 295 \end{bmatrix}$$

Example: Using Cayley Hamilton theorem find A^{-1} when $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

Solution:

The Characteristic equation of A is $|A - \lambda I| = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \text{ where}$$

$$S_1 = \text{sum of the main diagonal elements} = 1 + 1 + 1 = 3$$

$$S_2 = \text{Sum of the minors of the main diagonal elements.}$$

$$= \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$$

$$= (1 - 1) + (1 - 3) + (1 - 0)$$

$$= 0 - 2 + 1 = -1$$

$$S_3 = |A| = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= 1(1 - 1) - 0(2 + 1) + 3(-2 - 1)$$

$$= 0 - 0 + 3(-3) = -9$$

∴ The characteristic equation A is $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

By Cayley - Hamilton Theorem every square matrix satisfies its own Characteristic equation

$$\therefore A^3 - 3A^2 - A + 9I = O$$

$$A^{-1} = \frac{-1}{9} [A^2 - 3A - I] \quad \dots (1)$$

$$A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+3 & 0+0-3 & 3+0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0-1-1 & 3+1+1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$-3A = \begin{bmatrix} -3 & 0 & -9 \\ -6 & -3 & 3 \\ -3 & 3 & -3 \end{bmatrix}$$

$$(1) \Rightarrow A^{-1} = \frac{-1}{9} \left[\begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix} + \begin{pmatrix} -3 & 0 & -9 \\ -6 & -3 & 3 \\ -3 & 3 & -3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]$$

$$= \frac{-1}{9} \begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

Example: Verify Cayley- Hamilton for the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

Solution :

$$\text{Given } A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

The characteristic equation A is $|A - \lambda I| = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \dots (1) \quad \text{where}$$

$S_1 =$ Sum of the main diagonal elements

$$= 1 + 2 + 1 = 4$$

$S_2 =$ Sum of the minors of its leading diagonal elements

$$\begin{aligned} &= \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} \\ &= (2 - 6) + (1 - 7) + (2 - 12) \\ &= -4 - 6 - 10 = -20 \end{aligned}$$

$$\begin{aligned} S_3 &= |A| = \begin{vmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{vmatrix} \\ &= 1(2 - 6) - 3(4 - 3) + 7(8 - 2) \\ &= -4 - 3(1) + 7(6) \\ &= -4 - 3 + 42 = 35 \end{aligned}$$

$$\therefore (1) \Rightarrow \lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$$

By Cayley-Hamilton theorem

$$(2) \Rightarrow A^3 - 4A^2 - 20A - 35I = 0$$

To find A^2 and A^3 :

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 + 12 + 7 & 3 + 6 + 14 & 7 + 9 + 7 \\ 4 + 8 + 3 & 12 + 4 + 6 & 28 + 6 + 3 \\ 1 + 8 + 1 & 3 + 4 + 2 & 7 + 6 + 1 \end{bmatrix} \\ &= \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \end{aligned}$$

$$A^3 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{aligned}
 &= \begin{bmatrix} 20 + 92 + 23 & 60 + 46 + 46 & 140 + 69 + 23 \\ 15 + 88 + 37 & 45 + 44 + 74 & 105 + 66 + 37 \\ 10 + 36 + 14 & 30 + 18 + 28 + & 70 + 27 + 14 \end{bmatrix} \\
 &= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}
 \end{aligned}$$

$$A^3 - 4A^2 - 20A - 35I$$

$$\begin{aligned}
 &= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4 \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20 \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - \\
 &35 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} + \begin{bmatrix} -80 & -92 & -92 \\ -60 & -88 & -148 \\ -40 & -36 & -56 \end{bmatrix} + \begin{bmatrix} -20 & -60 & -140 \\ -80 & -40 & -60 \\ -20 & -40 & -20 \end{bmatrix} \\
 &\quad + \begin{bmatrix} -35 & 0 & 0 \\ 0 & -35 & 0 \\ 0 & 0 & -35 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

∴ The given matrix A satisfies its own characteristic equation.

Hence, Cayley-Hamilton theorem is verified.