3.2 LAPLACE TRANSFORM OF DERIVATIVES AND INTEGRALS

Problems using the formula

$$L[tf(t)] = \frac{-d}{ds} L[f(t)]$$

Example: Find the Laplace transform for $tsin4t$

Solution:

$$L[tsin4t] = \frac{-d}{ds} L[tsin4t]$$

$$= \frac{-d}{ds}\left[\frac{4}{s^2+4}\right]$$

$$= \frac{-[(s^2+16)0-4(2s)]}{(s^2+16)^2}$$

$$\therefore L[tsin4t] = \frac{8s}{(s^2+16)^2}$$

Example: Find $L[tsin^2t]$

Solution:

$$L[tsin^2t] = \frac{-d}{ds} L[\sin^2 t] = \frac{-d}{ds} L\left[\frac{(1-cos2t)}{2}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}[L(1) - L(cos2t)]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{s^2+4-s^2}{s(s^2+4)}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{4}{s(s^2+4)}\right]$$

$$= -\frac{4}{2}\frac{d}{ds}\left[\frac{1}{s(s^2+4)}\right]$$

$$= -2\left[\frac{0-(3s^2+4)}{(s^3+4s)^2}\right]$$

$$\therefore L[tsin^2t] = \frac{2(3s^2+4)}{(s^3+4s)^2}$$

Example: Find $L[tcos^2 2t]$

Solution:

$$L[cos^2 2t] = \frac{-d}{ds} L[\cos^2 2t] = \frac{-d}{ds} L\left[\frac{(1+cos4t)}{2}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}[L(1) + L(cos4t)]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{1}{s} + \frac{s}{s^2+16}\right]$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2} + \frac{(s^2+16)1-s.2s}{(s^2+16)^2}\right]$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2} + \frac{\boldsymbol{s^2+16-2s^2}}{\boldsymbol{(s^2+16)^2}}\right]$$

$$\therefore L[cos^2 2t] = \frac{1}{2}\left[\frac{1}{s^2} - \frac{\boldsymbol{16-s^2}}{\boldsymbol{(s^2+16)^2}}\right]$$

Example: Find the Laplace transform for $\boldsymbol{tsinh2t}$

Solution:

$$L[sinh2t] = \frac{-d}{ds}L[sinh2t]$$

$$= \frac{-d}{ds}\left[\frac{2}{s^2-4}\right]$$

$$= \frac{-[(s^2-4)0-2(2s)]}{(s^2-4)^2}$$

$$\therefore L[tsinh2t] = \frac{4s}{(s^2-4)^2}$$

Exampl Find the Laplace transform for $\boldsymbol{f(t) = sinat - atcosat}$

Solution:

$$L[sinat - atcosat] = \mathrm{L}(\sin at) - a\,\mathrm{L(tcosat)}$$

$$= \frac{a}{s^2+a^2} - a\left(\frac{-d}{ds}L[cosat]\right)$$

$$= \frac{a}{s^2+a^2} + a\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= \frac{a}{s^2+a^2} + a\left[\frac{(s^2+a^2)1-s(2s)}{(s^2+a^2)^2}\right]$$

$$= \frac{a}{s^2+a^2} + a\left[\frac{s^2+a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{a}{s^2+a^2} + a\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{a(s^2+a^2)+a(a^2-s^2)}{(s^2+a^2)^2}$$

$$= \frac{as^2+a^3+a^3-as^2)}{(s^2+a^2)^2}$$

$$\therefore L[sinat - atcosat] = \frac{2a^3}{(s^2+a^2)^2}$$

Example: Find the Laplace transform for the following

(i) $\boldsymbol{te^{-3t}sin2t}$ (ii) $\boldsymbol{te^{-t}cosat}$ 9iii) $\boldsymbol{tsinhtcos2t}$

Solution:

(i) $L[te^{-3t}sin2t] = L[tsin2t]_{s\to s+3} = \frac{-d}{ds}L[sin2t]_{s\to s+3}$

$$= \frac{-d}{ds}\left(\frac{2}{s^2+2^2}\right)_{s\to s+3}$$

$$= \left[\frac{(s^2+4)0-2(2s)}{(s^2+4)^2}\right]_{s\to s+3}$$

$$= \left[\frac{4s}{(s^2+4)^2}\right]_{s\to s+3}$$

$$\therefore\ L[te^{-3t}sin2t] = \frac{4(s+3)}{((s+3)^2+4\,)^2}$$

(ii) $L[te^{-t}cosat]\ = L[tcosat]_{s\to s+1} = \frac{-d}{ds}L[cosat]_{s\to s+1}$

$$= \frac{-d}{ds}\left(\frac{s}{s^2+a^2}\right)_{s\to s+1}$$

$$= -\left[\frac{(s^2+a^2)1-s(2s)}{(s^2+a^2)^2}\right]_{s\to s+1}$$

$$= -\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]_{s\to s+1}$$

$$= \left[\frac{s^2-a^2}{(s^2+a^2)^2}\right]_{s\to s+1}$$

$$\therefore\ L[te^{-t}cosat] = \frac{(s+1)^2-a^2}{((s+1)^2+a^2\,)^2}$$

(iii) $L[tsinhtcos2t]$

$$L[tsinhtcos2t] = L\left[t\left(\frac{e^t-e^{-t}}{2}\right)cos2t\right]$$

$$= \frac{1}{2}[L(te^t cos2t) - L(te^{-t}cos2t)]$$

$$= \frac{1}{2}\left[\frac{-d}{ds}L[cos2t]_{s\to s-1} + \frac{d}{ds}L[cos2t]_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[\frac{-d}{ds}\left(\frac{s}{s^2+4}\right)_{s\to s-1} + \frac{d}{ds}\left(\frac{s}{s^2+4}\right)_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[-\left[\frac{(s^2+4)1-s(2s)}{(s^2+4)^2}\right]_{s\to s-1} + \left[\frac{(s^2+4)1-s(2s)}{(s^2+4)^2}\right]_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[-\left[\frac{4-s^2}{(s^2+4)^2}\right]_{s\to s-1} + \left[\frac{4-s^2}{(s^2+4)^2}\right]_{s\to s+1}\right]$$

$$\therefore L[tsinhtcos2t] = \frac{1}{2}\left[\frac{(s-1)^2-4}{((s-1)^2+4\,)^2} + \frac{4-(s+1)^2}{((s+1)^2+4\,)^2}\right]$$

Problems using the formula

$$\boldsymbol{L[t^2 f(t)] = \frac{d^2}{ds^2}L[f(t)]}$$

Example:Find the Laplace transform for (i) $t^2 sint$ (ii) $t^2 cos2t$

Solution:

(i) $L[t^2 sint] = \frac{d^2}{ds^2}L[sint]$

$$= \frac{d^2}{ds^2}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{d}{ds}\left(\frac{[(s^2+1)0-1(2s)]}{(s^2+1)^2}\right)$$

$$= \frac{d}{ds}\left(\frac{-2s}{(s^2+1)^2}\right)$$

$$= -2\frac{d}{ds}\left(\frac{s}{(s^2+1)^2}\right)$$

$$= \frac{-2\left[(s^2+1)^2(1)-s(2)(s^2+1)(2s)\right]}{(s^2+1)^4}$$

$$= \frac{-2(s^2+1)\left[(s^2+1)-4s^2\right]}{(s^2+1)^4}$$

$$= \frac{-2\left[1-3s^2\right]}{(s^2+1)^3}$$

$$\therefore L[t^2 sint] = \frac{6s^2-2}{(s^2+1)^3}$$

(ii) $L[t^2cos2t] = \frac{d^2}{ds^2}L[cos2t]$

$$= \frac{d^2}{ds^2}\left[\frac{s}{s^2+4}\right]$$

$$= \frac{d}{ds}\left(\frac{\left[(s^2+4)1-s(2s)\right]}{(s^2+4)^2}\right)$$

$$= \frac{d}{ds}\left(\frac{4-s^2}{(s^2+4)^2}\right)$$

$$= \frac{\left[(s^2+4)^2(-2s)-(4-s^2)2(s^2+4)(2s)\right]}{(s^2+4)^4}$$

$$= \frac{2s(s^2+4)\left[(s^2+4)(-1)-(4-s^2)2\right]}{(s^2+4)^4}$$

$$= \frac{2s\left[s^2-12\right]}{(s^2+4)^3}$$

$$\therefore L[t^2 \cos 2t] = \frac{2s\left[s^2-12\right]}{(s^2+4)^3}$$

Example: Find the Laplace transform for (i) $t^2e^{-2t}cost$ (ii) $t^2e^{4t}sin3t$

Solution:

(i) $L[t^2e^{-2t}cost] = L[t^2cost]_{s\to s+2} = \frac{d^2}{ds^2}L[cost]_{s\to s+2}$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)_{s\to s+2}$$

$$= \frac{d}{ds}\left[\frac{(s^2+1)1-s(2s)}{(s^2+1)^2}\right]_{s\to s+2}$$

$$= \frac{d}{ds}\left[\frac{1-s^2}{(s^2+1)^2}\right]_{s\to s+2}$$

$$= \left[\frac{\left[(s^2+1)^2(-2s)-(1-s^2)2(s^2+1)(2s)\right]}{(s^2+1)^4}\right]_{s\to s+2}$$

$$= (s^2+1)\left[\frac{\left[(s^2+1)(-2s)-4s(1-s^2)\right]}{(s^2+1)^4}\right]_{s\to s+2}$$

$$= \left[\frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3}\right]_{s\to s+2}$$

$$= \left[\frac{2s^3-6s}{(s^2+1)^3}\right]_{s\to s+2}$$

$$\therefore \ L[t^2e^{-2t}cost] = \frac{2(s+2)^3-6(s+2)}{((s+2)^2+1)^3}$$

(ii)$L[t^2e^{4t}sin3t] = L[t^2sin3t]_{s\to s-4} = \frac{d^2}{ds^2}L[sin3t]_{s\to s-4}$

$$= \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)_{s\to s-4}$$

$$= \frac{d}{ds}\left[\frac{(s^2+9)0-3(2s)}{(s^2+9)^2}\right]_{s\to s-4}$$

$$= \frac{d}{ds}\left[\frac{-6s}{(s^2+9)^2}\right]_{s\to s-4} = -6\frac{d}{ds}\left[\frac{s}{(s^2+9)^2}\right]_{s\to s-4}$$

$$= -6\left[\frac{\left[(s^2+9)^2(1)-(s)2(s^2+9)(2s)\right]}{(s^2+9)^4}\right]_{s\to s-4}$$

$$= -6(s^2+9)\left[\frac{\left[(s^2+9)-4s^2\right]}{(s^2+9)^4}\right]_{s\to s-4}$$

$$= -6\left[\frac{9-3s^2}{(s^2+9)^3}\right]_{s\to s-4}$$

$$= \left[\frac{18s^2-54}{(s^2+9)^3}\right]_{s\to s-4}$$

$$\therefore \ L[t^2e^{4t}sin3t] = \frac{18(s-4)^2-54}{((s-4)^2+9)^3}$$

Problems using the formula

$$\boldsymbol{L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]ds}$$

This formula is valid if $\lim\limits_{t\to 0}\frac{f(t)}{t}$ is finite.

The following formula is very useful in this section

$$\int\frac{ds}{s} = logs$$

$$\int\frac{ds}{s+a} = \log(s+a)$$

$$\int\frac{s\,ds}{s^2+a^2} = \frac{1}{2}\log(s^2+a^2)$$

$$\int\frac{a\,ds}{s^2+a^2} = \tan^{-1}\frac{s}{a}$$

Example: Find $\boldsymbol{L\left[\frac{cosat}{t}\right]}$

Solution:

$$\lim_{t\to 0}\frac{cosat}{t} = \frac{cosa(0)}{0} = \frac{1}{0} = \infty$$

$\therefore$ Laplace transform does not exists.

Example: Find $L\left[\frac{sinat}{t}\right]$

Solution:

$$\lim_{t\to 0}\frac{sinat}{t}=\frac{sina(0)}{0}=\frac{0}{0}$$

$$=\lim_{t\to 0}acosat \qquad \text{(by applying L−Hospital rule)}$$

$\lim_{t\to 0}acosat=acos0=a$, finite quantity.

Hence Laplace transform exists

$$L\left[\frac{sinat}{t}\right]=\int_s^{\infty}L[(sinat)]ds$$

$$=\int_s^{\infty}\frac{a}{s^2+a^2}ds$$

$$=\left[\tan^{-1}\frac{s}{a}\right]_s^{\infty}$$

$$=\left[\tan^{-1}\infty-\tan^{-1}\frac{s}{a}\right]$$

$$=\left[\frac{\pi}{2}-\tan^{-1}\frac{s}{a}\right]$$

$$\therefore L\left[\frac{sinat}{t}\right]=cot^{-1}\frac{s}{a}$$

Example: Find $L\left[\frac{sin^3t}{t}\right]$

Solution:

$$\frac{sin^3t}{t}=\frac{3sint-sin3t}{4t}$$

$$\lim_{t\to 0}\frac{sin^3t}{t}=\lim_{t\to 0}\frac{3sint-sin3t}{4t}$$

$$=\frac{0-0}{0}=\frac{0}{0} \qquad \text{(by applying L−Hospital rule)}$$

$$=\lim_{t\to 0}\frac{3sint-sin3t}{4t}=0$$

Hence Laplace transform exists

$$L\left[\frac{sin^3t}{t}\right]=L\left[\frac{3sint-sin3t}{4t}\right]$$

$$=\frac{1}{4}\int_s^{\infty}L[(3sint-sin3t)]ds$$

$$=\frac{1}{4}\int_s^{\infty}\left(3\frac{1}{s^2+1}-\frac{3}{s^2+9}\right)ds$$

$$=\frac{1}{4}\left[3\tan^{-1}s-\tan^{-1}\frac{s}{3}\right]_s^{\infty}$$

$$=\frac{1}{4}\left[3(\tan^{-1}\infty-\tan^{-1}s)-\left(\tan^{-1}\infty-\tan^{-1}\frac{s}{3}\right)\right]$$

$$=\frac{1}{4}\left[(\frac{\pi}{2}-\tan^{-1}s)-\left(\frac{\pi}{2}-\tan^{-1}\frac{s}{3}\right)\ \right]$$

$$= \frac{1}{4}\left[\cot^{-1} s - cot^{-1}\frac{s}{3}\right]$$

Example: Find $L\left[e^{-2t}\frac{sin2tcos3t}{t}\right]$

Solution:

$$L\left[e^{-2t}\frac{sin2tcos3t}{t}\right] = L\left[\frac{sin2tcos3t}{t}\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\int_s^\infty L(\sin(3t+2t) - \sin(3t-2t))ds\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\int_s^\infty L((\sin 5t) - L(\sin t))ds\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\int_s^\infty \left[\frac{5}{s^2+5^2} - \frac{1}{s^2+1^2}\right]\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\left[\tan^{-1}\frac{s}{5} - \tan^{-1} s\right]_s^\infty\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\left[\left(\tan^{-1}\infty - \tan^{-1}\frac{s}{5}\right) - (\tan^{-1}\infty - \tan^{-1} s)\right]\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \tan^{-1}\frac{s}{5}\right) - \left(\frac{\pi}{2} - \tan^{-1} s\right)\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\cot^{-1}\frac{s}{5} - cot^{-1} s\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\cot^{-1}\frac{(s+2)}{5} - cot^{-1}(s+2)\right]$$

Example: Find the Laplace transform for $\frac{e^{-at}-e^{-bt}}{t}$

Solution:

$$\lim_{t\to 0}\frac{e^{-at}-e^{-bt}}{t} = \lim_{t\to 0}\frac{e^0-e^0}{0} = \frac{1-1}{0} = \frac{0}{0}$$ (use L− Hospital rule)

$$= \lim_{t\to 0}\frac{-ae^{-at}+be^{-bt}}{1}$$

$= -a + b = b - a =$ a finite quantity

Hence Laplace transform exists.

$$L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \int_s^\infty L[e^{-at} - e^{-bt}]ds$$

$$= \int_s^\infty [L(e^{-at}) - L(e^{-bt})]ds$$

$$= \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right)ds$$

$$= [\log(s+a) - \log(s+b)]_s^\infty$$

$$= \left[log\frac{s+a}{s+b}\right]_s^\infty$$

$$= \left[log\frac{s\left(1+\frac{a}{s}\right)}{s\left(1+\frac{b}{s}\right)}\right]_s^\infty$$

$$= log1 - log\frac{s+a}{s+b} = 0 - log\frac{s+a}{s+b} \qquad \because log1 = 0$$

$$= log\frac{s+a}{s+b}$$

Example: Find the Laplace transform of $\frac{1-cost}{t}$

Solution:

$$\lim_{t\to 0}\frac{1-cost}{t} = \frac{0}{0} \qquad \lim_{t\to 0}\frac{sint}{1} = \frac{0}{1} = 0 \qquad \text{(use L– Hospital rule)}$$

$L\left[\frac{1-cost}{t}\right]$ exists.

$$L\left[\frac{1-cost}{t}\right] = \int_s^\infty L[(1-cost)]ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right)ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \left[\log s - \log\sqrt{s^2+1}\right]_s^\infty$$

$$= \left[log\frac{s}{\sqrt{s^2+1}}\right]_s^\infty$$

$$= 0 - log\frac{s}{\sqrt{s^2+1}}$$

$$= log\frac{\sqrt{s^2+1}}{s}$$

Example: Find the Laplace transform for $\frac{cosat-cosbt}{t}$

Solution:

$$\lim_{t\to 0}\frac{cosat-cosbt}{t} = \frac{1-1}{0} = \frac{0}{0} \qquad \text{(use L– Hospital rule)}$$

$$= \lim_{t\to 0}\frac{-asinat+bsinbt}{1} = 0 = \text{a finite quantity}$$

Hence Laplace transform exists.

$$L\left[\frac{cosat-cosbt}{t}\right] = \int_s^\infty L[cosat - cosbt]ds$$

$$= \int_s^\infty [L(cosat) - L(cosbt)]ds$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds$$

$$= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[log\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$$

$$= \frac{1}{2}\left[log\frac{s^2\left(1+\frac{a^2}{s^2}\right)}{s^2\left(1+\frac{b^2}{s^2}\right)}\right]_s^\infty$$

$$= \frac{1}{2}\left[log \frac{\left(1+\frac{a^2}{s^2}\right)}{\left(1+\frac{b^2}{s^2}\right)}\right]_s^\infty$$

$$= \frac{1}{2}\left[log1 - log\frac{s^2+a^2}{s^2+b^2}\right] = -\frac{1}{2}\left[log\frac{s^2+a^2}{s^2+b^2}\right] \qquad [\because log1 = 0]$$

$$= \frac{1}{2}\left[log\frac{s^2+b^2}{s^2+a^2}\right]$$

Example: Find the Laplace transform of $\frac{\boldsymbol{sin^2t}}{\boldsymbol{t}}$

Solution:

$$\frac{sin^2t}{t} = \frac{1-cos2t}{2t}$$

$$\lim_{t\to 0}\frac{\mathbf{1-cos2t}}{\mathbf{2t}} = \frac{0}{0}$$

$$\lim_{t\to 0}\frac{\mathbf{2sin2t}}{\mathbf{2}} = \frac{0}{1} = 0 \qquad \text{(use L} - \text{Hospital rule)}$$

Laplace transform exists.

$$L\left[\frac{sin^2t}{t}\right] = L\left[\frac{1-cos2t}{2t}\right] = \frac{1}{2}\int_s^\infty L[(1-cos2t)]ds$$

$$= \frac{1}{2}\int_s^\infty [L(1) - L(cos2t)]ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log s - \log\sqrt{s^2+4}\right]_s^\infty$$

$$= \frac{1}{2}\left[log\frac{s}{\sqrt{s^2+4}}\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - log\frac{s}{\sqrt{s^2+4}}\right]$$

$$= \frac{1}{2}log\frac{\sqrt{s^2+4}}{s}$$

Example: Find the Laplace transform for $\frac{\boldsymbol{sin2tsin5t}}{\boldsymbol{t}}$

Solution:

$$L\left[\frac{sin2tsin5t}{t}\right] = \int_s^\infty L[sin2tsin5t]ds$$

$$= \int_s^\infty \frac{1}{2}[L(\cos(-3t) - L(cos7t)]ds$$

$$= \frac{1}{2}\int_s^\infty [L(\cos(3t) - L(cos7t)]ds \qquad [\because \cos(-\theta) = cos\theta]$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{s}{s^2+9} - \frac{s}{s^2+49}\right)ds$$

$$= \frac{1}{2}\left[\frac{1}{2}\log(s^2+9) - \frac{1}{2}\log(s^2+49)\right]_s^\infty$$

$$= \frac{1}{4}\left[log\,\frac{s^2+9}{s^2+49}\right]_s^\infty$$

$$= \frac{1}{4}\left[log\,\frac{s^2\left(1+\frac{9}{s^2}\right)}{s^2\left(1+\frac{49}{s^2}\right)}\right]_s^\infty$$

$$= \frac{1}{4}\left[log\,\frac{\left(1+\frac{9}{s^2}\right)}{\left(1+\frac{49}{s^2}\right)}\right]_s^\infty$$

$$= \frac{1}{4}\left[log1 - log\,\frac{s^2+9}{s^2+49}\right] = -\frac{1}{4}\left[log\,\frac{s^2+9}{s^2+49}\right] \qquad [\because log1 = 0]$$

$$= \frac{1}{4}\left[log\,\frac{s^2+49}{s^2+9}\right]$$

Problems using $L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)]$

Example: Find the Laplace transform for (i) $\int_0^t e^{-2t}dt$ (ii) $\int_0^t cos2tdt$ (iii) $\int_0^t tsin3tdt$ (iv) $t\int_0^t costdt$

Solution:

(i) $L\left[\int_0^t e^{-2t}dt\right] = \frac{1}{s}L[e^{-2t}] = \frac{1}{s}\left(\frac{1}{s+2}\right)$

$$\therefore L\left[\int_0^t e^{-2t}dt\right] = \frac{1}{s(s+2)}$$

(ii) $L\left[\int_0^t cos2tdt\right] = \frac{1}{s}L[cos2t] = \frac{1}{s}\left(\frac{s}{s^2+4}\right)$

$$\therefore L\left[\int_0^t cos2tdt\right] = \frac{1}{s^2+4}$$

(iii) $L\left[\int_0^t tsin3tdt\right] = \frac{1}{s}L[tsin3t]$

$$= \frac{1}{s}\left[\frac{-d}{ds}[L[sin3t]]\right]$$

$$= \frac{-1}{s}\left[\frac{d}{ds}\left[\frac{3}{s^2+9}\right]\right]$$

$$= \frac{-1}{s}\left[\frac{-6s}{(s^2+9)^2}\right]$$

$$\therefore L\left[\int_0^t tsin3tdt\right] = \frac{6}{(s^2+9)^2}$$

(iv) $L\left[t\int_0^t costdt\right] = \frac{-d}{ds}L\left[\int_0^t costdt\right]$

$$= \frac{-d}{ds}\left[\frac{1}{s}\left(\frac{s}{s^2+1}\right)\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= -\left[\frac{-2s}{(s^2+1)^2}\right]$$

$$\therefore L\left[\int_0^t tsin3tdt\right] = \frac{2s}{(s^2+1)^2}$$

Example: Find the Laplace transform for $e^{-t}\int_0^t tcos4tdt$

Solution:

$$L\left[e^{-t}\int_0^t tcos4tdt\right] = L\left[\int_0^t tcos4tdt\right]_{s\to s+1} = \left[\frac{-1}{s}\frac{d}{ds}L(cos4t)\right]_{s\to s+1}$$

$$= -\left(\frac{1}{s}\frac{d}{ds}\frac{s}{s^2+16}\right)_{s\to s+1}$$

$$= \left[\frac{-1}{s}\ \frac{(s^2+16)1-s(2s)}{(s^2+16)^2}\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\frac{(s^2+16-2s^2)}{(s^2+16)^2}\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\frac{(-s^2+16)}{(s^2+16)^2}\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\frac{(s^2-16)}{(s^2+16)^2}\right]_{s\to s+1}$$

$$\therefore\ L\left[e^{-t}\int_0^t tcos4tdt\right] = \frac{1}{s+1}\left[\frac{(s+1)^2-16}{((s+1)^2+16\,)^2}\right]$$

Example: Find the Laplace transform of $e^{-t}\int_0^t \frac{sint}{t}dt$

Solution:

$$L\left[e^{-t}\int_0^t \frac{sint}{t}dt\right] = L\left[\int_0^t \frac{sint}{t}dt\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}L\left(\frac{sint}{t}\right)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\int_s^\infty L(sint)ds\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\ \int_s^\infty \frac{1}{s^2+1}\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\left[\tan^{-1}s\,\right]_s^\infty\ \right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\left(\tan^{-1}\infty - \tan^{-1}s\right)\ \right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\left(\frac{\pi}{2} - \tan^{-1}s\right)\ \ \right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\cot^{-1}s\ \right]_{s\to s+1}$$

$$\therefore L\left[e^{-t}\int_0^t \frac{sint}{t}dt\right] = \frac{1}{s+1}cot^{-1}(s+1)$$

Evaluation of integrals using Laplace transform

Note: (i) $\int_0^\infty f(t)e^{-st}dt = L[f(t)]$

(ii) $\int_0^\infty f(t)e^{-at}dt = \left[L[f(t)]\right]_{s=a}$

(iii) $\int_0^\infty f(t)dt = \left[L[f(t)]\right]_{s=0}$

Example: If $L[f(t)] = \frac{s+2}{s^2+4}$, then find the value of $\int_0^\infty f(t)dt$

Solution:

Given $L[f(t)] = \frac{s+2}{s^2+4}$

We know that $\int_0^\infty f(t)dt = [L[f(t)]]_{s=0}$

$$= \left[\frac{s+2}{s^2+4}\right]_{s=0} = \frac{2}{4}$$

$$\int_0^\infty f(t)dt = \frac{1}{2}$$

Example: If $L[f(t)] = \frac{5s+4}{s^2-9}$, then find the value of $\int_0^\infty e^{-2t}f(t)dt$

Solution:

Given $L[f(t)] = \frac{5s+4}{s^2-9}$

We know that $\int_0^\infty e^{-2t}f(t)dt = [L[f(t)]]_{s=2}$

$$= \left[\frac{5s+4}{s^2-9}\right]_{s=2} = \frac{14}{-5}$$

$$\therefore \int_0^\infty e^{-2t}f(t)dt = \frac{-14}{5}$$

Example:Find the values of the following integrals using Laplace transforms:

(i) $\int_0^\infty te^{-2t}cos2tdt$ **(ii)** $\int_0^\infty t^2e^{-t}sintdt$ **(iii)** $\int_0^\infty \left(\frac{e^{-t}-e^{-2t}}{t}\right)dt$

(iv) $\int_0^\infty \left(\frac{1-cost}{t}\right)e^{-t}dt$ **(v)** $\int_0^\infty \left(\frac{e^{-at}-cosbt}{t}\right)dt$

Solution:

(i) $\int_0^\infty te^{-2t}cos2tdt = L[tcos2t]_{s=2} = \left[\frac{-d}{ds}L(cos2t)\right]_{s=2}$

$$= \frac{-d}{ds}\left(\frac{s}{s^2+4}\right)_{s=2}$$

$$= -\left[\frac{(s^2+4)1-s(2s)}{(s^2+4)^2}\right]_{s=2}$$

$$= -\left[\frac{(4-s^2)}{(s^2+4)^2}\right]_{s=2}$$

$$= -\frac{(4-4)}{(4+4)^2} = 0$$

(ii) $\int_0^\infty t^2e^{-t}sintdt = L[t^2sint]_{s=1} = \frac{d^2}{ds^2}L[sint]_{s=1}$

$$= \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right)_{s=1}$$

$$= \frac{d}{ds}\left[\frac{-1(2s)}{(s^2+1)^2}\right]_{s=1}$$

$$= -2\frac{d}{ds}\left[\frac{s}{(s^2+1)^2}\right]_{s=1}$$

$$= -2\left[\frac{\left[(s^2+1)^2(1)-s.2(s^2+1)(2s))\right]}{(s^2+1)^4}\right]_{s=1}$$

$$= -2\left[\frac{\left[(s^2+1)[(s^2+1)-4s^2)]\right]}{(s^2+1)^4}\right]_{s=1}$$

$$= -2\left[\frac{(1-3s^2)}{(s^2+1)^3}\right]_{s=1}$$

$$= \left[\frac{6s^3-2}{(s^2+1)^3}\right]_{s=1} = \frac{4}{8} = \frac{1}{2}$$

(iii) $\int_0^\infty \left(\frac{e^{-t}-e^{-2t}}{t}\right)dt = L\left[\frac{e^{-t}-e^{-2t}}{t}\right]_{s=0} = \int_s^\infty [L[e^{-t}-e^{-2t}]ds]_{s=0}$

$$= \int_s^\infty \left[[L(e^{-t}) - L(e^{-2t})]\,ds\right]_{s=0}$$

$$= \int_s^\infty \left[\left(\frac{1}{s+1} - \frac{1}{s+2}\right)ds\right]_{s=0}$$

$$= \{[\log(s+1) - \log(s+2)]_s^\infty\}_{s=0}$$

$$= \left\{\left[log\frac{s+1}{s+2}\right]_s^\infty\right\}_{s=0}$$

$$= \left\{log\frac{s\left(1+\frac{1}{s}\right)}{s\left(1+\frac{2}{s}\right)}\Bigg|_s^\infty\right\}_{s=0}$$

$$= \left[0 - log\frac{s+1}{s+2}\right]_{s=0} \qquad \because log1 = 0$$

$$= \left[log\frac{s+2}{s+1}\right]_{s=0} = log2$$

(iv) $\int_0^\infty \left(\frac{1-cost}{t}\right)e^{-t}dt$

$$\int_0^\infty \left(\frac{1-cost}{t}\right)e^{-t}dt = L\left[\frac{1-cost}{t}\right]_{S=1} = \int_s^\infty [L[(1-cost)]ds]_{S=1}$$

$$= \int_s^\infty \left[[L(1) - L(cost)]ds\right]_{S=1}$$

$$= \int_s^\infty \left[\left(\frac{1}{s} - \frac{s}{s^2+1}\right)ds\right]_{S=1}$$

$$= \left\{\left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty\right\}_{S=1}$$

$$= \left\{\left[\log s - \log\sqrt{s^2+1}\right]_s^\infty\right\}_{S=1}$$

$$= \left\{\left[log\frac{s}{\sqrt{s^2+1}}\right]_s^\infty\right\}_{S=1}$$

$$= \left[0 - log\frac{s}{\sqrt{s^2+1}}\right]_{S=1}$$

$$= \left[log\frac{\sqrt{s^2+1}}{s}\right]_{S=1}$$

$$= log\sqrt{2}$$

(v) $\int_0^\infty \left(\frac{e^{-at}-cosbt}{t}\right) dt$

$$\int_0^\infty \left(\frac{e^{-at}-cosbt}{t}\right) dt = L\left[\frac{e^{-at}-cosbt}{t}\right]_{S=0} = \int_s^\infty [L[(e^{-at} - cosbt)]ds]_{S=0}$$

$$= \int_s^\infty \left[[L(e^{-at}) - L(cosbt)]ds\right]_{S=0}$$

$$= \int_s^\infty \left[\left(\frac{1}{s+a} - \frac{s}{s^2+b^2}\right) ds\right]_{S=0}$$

$$= \left\{\left[\log(s+a) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty\right\}_{S=0}$$

$$= \left\{\left[\log(s+a) - \log\sqrt{s^2+b^2}\right]_s^\infty\right\}_{S=0}$$

$$= \left\{\left[log\,\frac{s+a}{\sqrt{s^2+b^2}}\right]_s^\infty\right\}_{S=0}$$

$$= \left[0 - log\,\frac{s+a}{\sqrt{s^2+b^2}}\right]_{S=0}$$

$$= \left[log\,\frac{\sqrt{s^2+b^2}}{s+a}\right]_{S=0}$$

$$= log\,\frac{\sqrt{b^2}}{a}$$

$$= log\,\frac{b}{a}$$

3.2(a)PERIODIC FUNCTIONS

Definition: A function $f(t)$ is said to be periodic if $f(t+T) = f(t)$ for all values of t and for certain values of T. The smallest value of T for which $f(t+T) = f(t)$ for all t is called periodic function.

Example:

$sint = \sin(t+2\pi) = \sin(t+4\pi) \cdots$

$\therefore\ sint$ is periodic function with period 2π.

Let $f(t)$ be a periodic function with period T. Then

$$\boxed{L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt}$$

Problems on Laplace transform of Periodic function

Example:Find the Laplace transform of $f(t) = \begin{cases} sin\omega t; 0 < t < \frac{\pi}{\omega} \\ 0\,; \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$ $f\left(t + \frac{2\pi}{\omega}\right) = f(t)$

Solution:

The given function is a periodic function with period $T = \frac{2\pi}{\omega}$

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}} \left[\int_0^{\frac{\pi}{\omega}} sin\omega t e^{-st} dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-st}(0)dt \right]$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}} \int_0^{\frac{\pi}{\omega}} sin\omega t e^{-st} dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}} \left[\frac{e^{-st}}{(-s)^2+\omega^2} (-s sin\omega t - \omega cos\omega t) \right]_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}} \left\{ \frac{e^{\frac{-s\pi}{\omega}}}{s^2+\omega^2} [-s sin\pi - \omega cos\pi] + \frac{\omega}{s^2+\omega^2} \right\}$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}} \left[\frac{e^{\frac{-s\pi}{\omega}}\omega+\omega}{s^2+\omega^2} \right]$$

$$= \frac{1}{1^2-\left(e^{\frac{-\pi s}{\omega}}\right)^2} \left[\frac{\omega\left(e^{\frac{-s\pi}{\omega}}+1\right)}{s^2+\omega^2} \right]$$

$$= \frac{1}{\left(1-e^{\frac{-\pi s}{\omega}}\right)\left(1+e^{\frac{-\pi s}{\omega}}\right)} \left[\frac{\omega\left(e^{\frac{-s\pi}{\omega}}+1\right)}{s^2+\omega^2} \right]$$

$$\therefore L[f(t)] = \frac{\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)(s^2+\omega^2)}$$

Example: Find the Laplace transform of $\boldsymbol{f(t) = \begin{cases} E; 0 \le t \le a \\ -E ; a \le t \le 2a \end{cases}}$ given that $\boldsymbol{f(t+2a) = f(t)}$.

Solution:

The given function is a periodic function with period $T = 2a$

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a Ee^{-st}dt + \int_a^{2a} -Ee^{-st}dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[E\int_0^a e^{-st}dt - E\int_a^{2a} e^{-st}dt \right]$$

$$= \frac{E}{1-e^{-2as}} \left[\left[\frac{e^{-st}}{-s}\right]_0^a - \left[\frac{e^{-st}}{-s}\right]_a^{2a} \right]$$

$$= \frac{E}{1-e^{-2as}} \left[\frac{e^{-as}}{-s} + \frac{1}{s} - \frac{e^{-2as}}{s} - \frac{e^{-as}}{s} \right]$$

$$= \frac{E}{1-e^{-2as}} \left[\frac{1-2e^{-as}+e^{-2as}}{s} \right]$$

$$= \frac{E}{1^2-(e^{-as})^2} \left[\frac{(1-e^{-as})^2}{s} \right]$$

$$= \frac{E}{(1-e^{-as})(1+e^{-as})}\left[\frac{(1-e^{-as})^2}{s}\right]$$

$$= \frac{E}{s}\frac{(1-e^{-as})}{(1+e^{-as})}$$

$$\therefore L[f(t)] = \frac{E}{s}\tanh\left(\frac{as}{2}\right)$$

Example: Find the Laplace transform of $f(t) = \begin{cases} 1; 0 \le t \le \frac{a}{2} \\ -1\,; \frac{a}{2} \le t \le a \end{cases}$ **given that** $f(t+a) = f(t)$.

Solution:

The given function is a periodic function with period $T = a$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{\frac{a}{2}}(1)e^{-st}dt + \int_{\frac{a}{2}}^{a}(-1)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{\frac{a}{2}} e^{-st}dt - \int_{\frac{a}{2}}^{a} e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\left[\frac{e^{-st}}{-s}\right]_0^{\frac{a}{2}} - \left[\frac{e^{-st}}{-s}\right]_{\frac{a}{2}}^{2a}\right]$$

$$= \frac{1}{1-e^{-as}}\left[\frac{e^{\frac{-sa}{2}}}{-s} + \frac{1}{s} + \frac{e^{-as}}{s} - \frac{e^{\frac{-sa}{2}}}{s}\right]$$

$$= \frac{1}{1-e^{-as}}\left[\frac{1-2e^{\frac{-sa}{2}}+e^{-as}}{s}\right]$$

$$= \frac{1}{1^2-\left(e^{\frac{-sa}{2}}\right)^2}\left[\frac{\left(1-e^{\frac{-sa}{2}}\right)^2}{s}\right]$$

$$= \frac{1}{\left(1-e^{\frac{-sa}{2}}\right)\left(1+e^{\frac{-sa}{2}}\right)}\left[\frac{\left(1-e^{\frac{-sa}{2}}\right)^2}{s}\right]$$

$$= \frac{1}{s}\frac{\left(1-e^{\frac{-sa}{2}}\right)}{\left(1+e^{\frac{-sa}{2}}\right)} \qquad \left[\because tanhx = \frac{(1-e^{-2x})}{(1+e^{-2x})}\right]$$

$$\therefore L[f(t)] = \frac{1}{s}\tanh\left(\frac{as}{4}\right)$$

Example: Find the Laplace transform of $f(t) = \begin{cases} t; 0 \le t \le a \\ 2a-t\,; a \le t \le 2a \end{cases}$ **given that** $f(t+2a) = f(t)$.

Solution:

The given function is a periodic function with period $T = 2a$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st}f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a te^{-st}dt + \int_a^{2a}(2a-t)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[t\left(\frac{e^{-st}}{-s}\right) - \left(\frac{e^{-st}}{(-s)^2}\right)\right]_0^a - \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{(-s)^2}\right)\right]_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1-2e^{-as}+e^{-2as}}{s^2}\right]$$

$$= \frac{1}{1^2-(e^{-as})^2}\left[\frac{(1-e^{-as})^2}{s^2}\right]$$

$$= \frac{1}{(1-e^{-as})(1+e^{-as})}\left[\frac{(1-e^{-as})^2}{s^2}\right]$$

$$= \frac{1}{s^2}\frac{(1-e^{-as})}{(1+e^{-as})}$$

$$= \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$