UNIT I INTRODUCTION CO ORDINATE SYSTEMS AND COORDINATE SYSTEMS AND TRANSFORMATION

INTRODUCTION :

In general ,the physical quantity with in EM (electro magnetic) are functions of space and time. Coordinate system is required to describe the spatial variations of the quantities.

A point or vector can be represented in any curvilinear coordinate system , which may be orthogonal or non_orthogonal.

An orthogonal system is one in which the coordinates are mutually perpendicular.

Non orthogonal systems are hard to work and they are of no practical use.

Example of orthogonal coordinate systems are the Cartesian or Rectangular, the Circular cylindrical , the spherical ,the elliptic cylindrical, the parabolic cylindrical, the Conical ,the Prolate spheroidal, the oblate spheroidaland the ellipsoidal.

SCALARS AND VECTORS:

A scalar is a quantity like mass or temperature that only has a magnitude. On the other hand, a vector is a mathematical object that has magnitude and direction. A line of given length and pointing along a given direction, such as an arrow, is the typical representation of a vector. Typical notation to designate a vector is a boldfaced character, a character with and arrow on it, or a character with a line under it (i.e., $\mathbf{A}, \vec{A}, \text{or } \underline{A}$). The magnitude of a vector is its length and is normally denoted by $|\mathbf{A}|$ or A.

In EM theory field is a function that specifies a particular quantity everywhere in a region. The field is said to be Scalar (or Vector)field. Example for scalar field temperature distribution in a building and sound intensity in a theater etc. Example for vector field gravitational force on a body in space and velocity of raindrops in atmosphere etc.

UNIT VECTOR

A unit vector is a vector of unit length. A unit vector is sometimes denoted by replacing the arrow on a vector with a "^" or just adding a "^" on a boldfaced character (i.e., $\hat{e}, \text{or } \hat{\mathbf{e}}$). Therefore, $|\hat{\mathbf{e}}| = 1$

Any vector can be made into a unit vector by dividing it by its length.

$$\hat{\mathbf{e}} = \frac{\mathbf{u}}{|\mathbf{u}|}$$

Any vector can be fully represented by providing its magnitude and a unit vector along its direction.

$$\mathbf{u} = u\hat{\mathbf{e}}$$

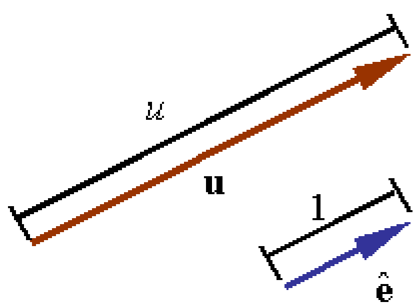
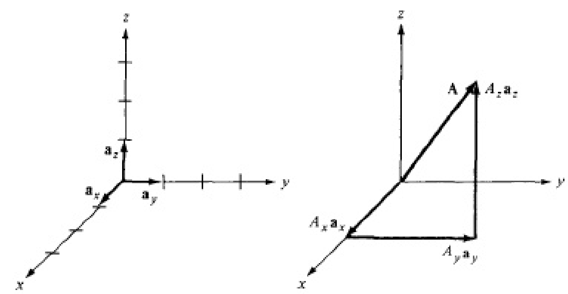


Fig unit vectors $\hat{e}$



A vector **A** in Cartesian coordinate may be represented as (A_x, A_y, A_z) or $A_x a_x + A_y a_y + A_z a_z$.

Magnitude of Vector A is $A = \sqrt{(A_x{}^2 + A_y{}^2 + A_z{}^2)}$

Unit vector $a_A = \dfrac{A_x a_x + A_y a_y + A_z a_z}{\sqrt{(A_x{}^2 + A_y{}^2 + A_z{}^2)}}$

Fig 1a : Unit vectors a_x,a_y and a_z Fig (b) : component of A along a_x,a_y and a_z

VECTOR ADDITION AND SUBTRACTION :

Two vectors A and B can be added together to give another vector C. that is

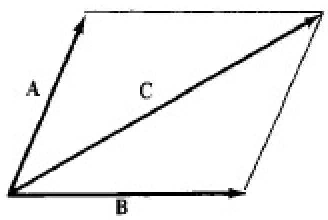
C=A+ B

The vector addition is carried out component by component . If A= (A_x, A_y, A_z) and B = (B_x ,B_y , B_z)

$$C = (A_x + B_x)\, a_x + \left(A_y + B_y\right) a_y + (A_z + B_z) a_z$$

Vector Subtraction is carried out as D = A – B = A + (-B)

$$D = (A_x - B_x)\, a_x + \left(A_y - B_y\right) a_y + (A_z - B_z) a_z$$



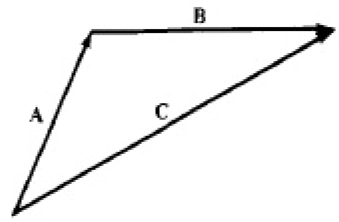
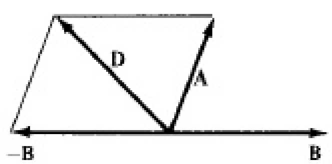


Fig2a: Vector addition C = A + B parallelogram rule ; Fig b: head –to – tail rule



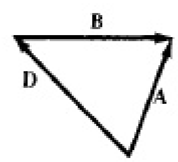


Fig 3a: Vector subtraction D= A- B parallelogram rule Figb : head –to – tail rule

Basic laws

Commutative A+ B = B + A

Associative A + (B + C) = (A + B) + C

Distributive k(A + B) = kA + kB

Where k is a scalar.

VECTOR MULTIPLICATION :

When two vectors A and B are multiplied the result is either scalar or vector depending on the method of multiplication. There are two types of vector multiplication

1. Scalar (dot) product : A. B
2. Vector (cross) product : A X B

Multiplication of three vectors A,B and C

3. Scalar triple product : A.(B X C)
4. Vector triple product : A X (B X C)

DOT PRODUCT

The dot product of two vectors A and B is written as A .B , is defined as the product of the magnitudes of A and B and Cosine of angle between them.

Mathematically expressed as $A.B = AB \cos \theta_{AB}$

Where θ_{AB} → angle between A and B

The result of product must be a scalar. If A= (A_x, A_y, A_z) and B = (B_x ,B_y , B_z) the dot product is obtained by multiplying A and B component by component.

$$A.B = A_xB_x + A_yB_y + A_zB_z$$

If two vectors A and B are orthogonal (perpendicular) with each other ,

A.B = 0

Commutative A.B = B .A

Distributive A.(B+ C) = A.B + A.C

$$A.A = |A|^2 = A^2$$

Similarly $a_x . a_y = a_y . a_z = a_z . a_x = 0$

$a_x . a_x = a_y . a_y = a_z . a_z = 1$

CROSS PRODUCT :

The cross product of two vectors A and B written as A X B , is a vector quantity whose magnitude is the area of the parallelepiped formed by A and B and is in the direction of a right - handed screw as A is turned into B.

$$A\ X\ B = AB\ sin\theta_{AB}a_n$$

a_n → unit vector normal to the plane containing A and B.

The cross product is also called as vector product because the result is a vector. If A= (A_x, A_y, A_z) and B = (B_x ,B_y , B_z) then

$$A\,X\,B = \begin{vmatrix} a_x & a_y & a_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$= (A_yB_z - A_zB_y)\,a_x + (A_zB_x - A_xB_z)\,a_y + (A_xB_y - A_yB_x)\,a_z$

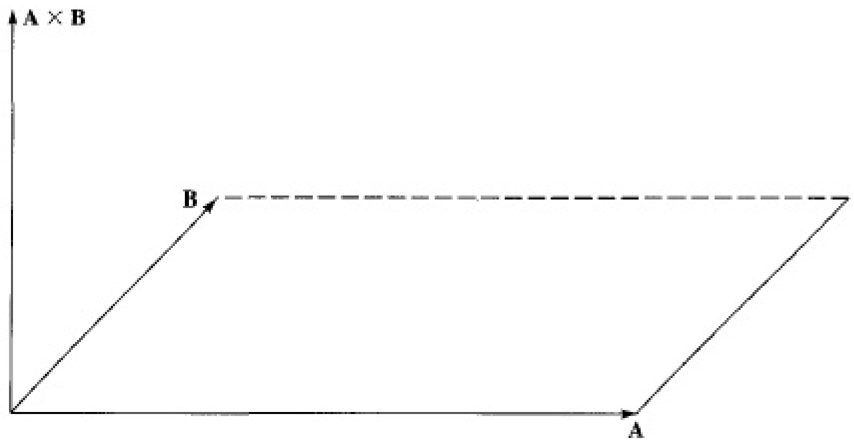
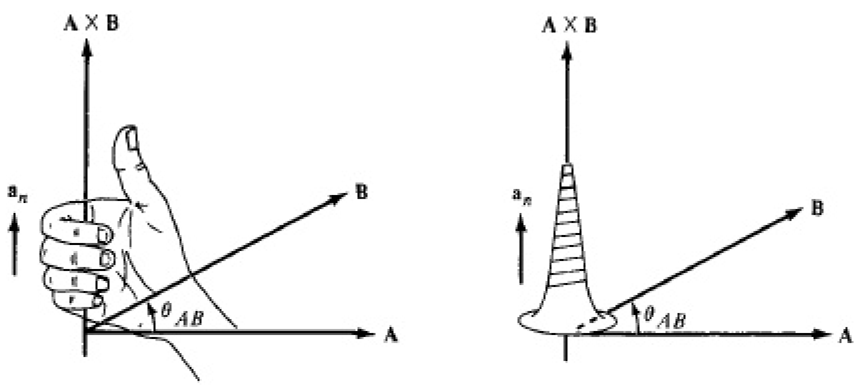


Fig 4: The cross product of A and B is a Vector with magnitude equal to the area of the parallelogram and direction as indicated



Right hand rule Right handed screw rule

Fig5 : Direction of A X B and a_n using Right hand rule and Right handed screw rule.

Properties of cross product :

i) Not Commutative $A\,X\,B \neq B\,XA$

Anti Commutative $A\,X\,B = -B\,XA$

ii) Not associative : $A\,X(B\,X\,C) \neq (A\,X\,B)X\,C$

iii) Distributive $A\,X(\,B + C) = \,AX\,B + A\,X\,C$

iv) A X A =0

also a_x X a_y = a_z , a_y X a_z = a_x , a_z X a_x = a_y

SCALAR TRIPLE PRODUCT

The scalar triple product of three vector A,B,C is given as

$$A.(B\,X\,C\,) = \,B.(\,C\,X\,A\,) = C.(\,A\,X\,B)$$

If A= (A_x, A_y, A_z) , B = (B_x ,B_y , B_z) and C = (C_x , C_y ,C_z)

$$A.(B\,X\,C\,) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

The result of this vector multiplication is scalar hence it is called as scalar triple product.

VECTOR TRIPLE PRODUCT

The vector triple product of three vector A,B,C is given as

$$A\,X(\,B\,X\,C\,) = \,B(\,A\,.C) - \,C\,(\,A\,.B\,)$$

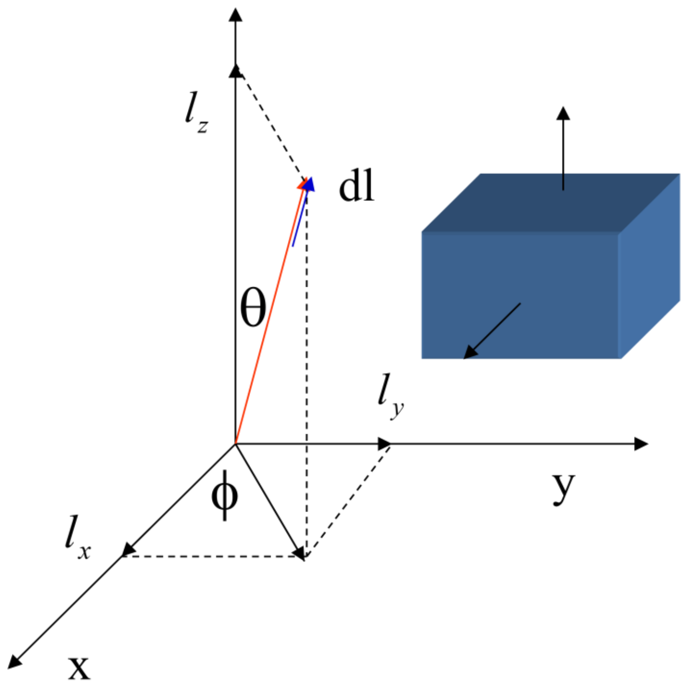
$$(\,A.B\,)C\,\neq A\,(\,B\,.C)$$

$$(\,A\,.B\,)C = C(\,A.B)$$

CARTESIAN COORDINATES (X,Y,Z):

A Point P can be represented as (x,y,z)

$$d\vec{l} = dl_x\,\hat{x} + dl_y\,\hat{y} + dl_z\,\hat{z} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$



$$d\vec{s}_x = \hat{x}\, dl_y dl_z = \hat{x}\, dydz$$

$$d\vec{s}_y = \hat{y}\, dl_x dl_z = \hat{y}\, dxdz$$

$$d\vec{s}_z = \hat{z}\, dl_x dl_y = \hat{z}\, dxdy$$

$$dv = dxdydz$$

The ranges of Cartesian co-ordinate variables are $-\infty < X < \infty$

$-\infty < Y < \infty$

$-\infty < Z < \infty$

Fig 6: Cartesian co-ordinate

A ... itten as $A = A_x a_x$

a_x, a_x, a_z → unit vectors in x,y,z directions.

Circular or Cylindrical coordinates (ρ (or) r, Φ,z):

A point P in Cylindrical coordinates is represented as (ρ (or) r , Φ,z)

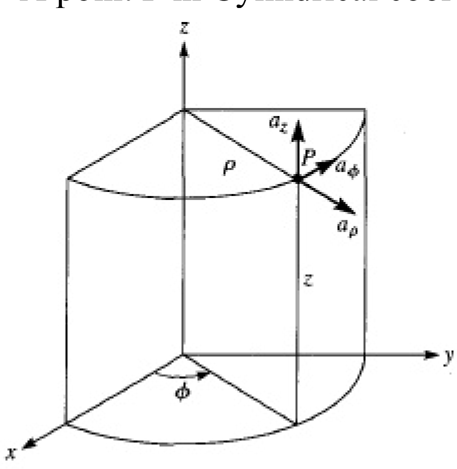


Fig 7 : A point P and unit vectors in cylindrical co-ordinate system

$\overrightarrow{\rho}\ X\ \overrightarrow{\varphi} = \vec{z}$ $\quad \vec{\varphi}\ X\ \vec{z} = \vec{\rho}$

$\vec{\varphi}\ X\ \vec{z} = \vec{\rho}$

$\hat{\rho} \times \hat{\rho} = 0$ $\quad \hat{\rho} \bullet \hat{\rho} = 1$

A vector A in cylindrical coordinate is

$$\vec{A} = \hat{\rho} A\rho + \hat{\phi} A_\phi + \hat{z} A_z$$

Magnitude of vector A is

$$\left|\vec{A}\right| = \sqrt{A_r^2 + A_\phi^2 + A_z^2}$$

$$dv = \rho d\phi d\rho dz$$

Differential surface

$\overrightarrow{ds_z} = \rho\, d\Phi dr\, \vec{z}\ ,\ \overrightarrow{ds_r} = \rho d\Phi dz\, \vec{r}\ ,\ \overrightarrow{ds_\Phi} = drdz\, \vec{\Phi}$

ρ (or) r → radius of the cylinder passing through P

Φ → azimuth angle measured from the x – axis in xy – plane

Z → same as in Cartesian system

The ranges of the variables are $0 \le \rho \le \infty$, $0 \le \Phi \le 2\pi$, $-\infty < Z < \infty$

Unit vector relations

aρ, aφ, az are mutually perpendicular because the coordinate system is orthogonal. $a\rho\,.\,a\rho = a\varphi\,.\,a\varphi = az\,.\,az = 1$

$$a\rho\,.\,a\varphi = a\varphi\,.\,az = az\,.\,a\rho = 0$$

$a\rho\ X\ a\varphi = az$

$a\varphi\ X\ az = a\rho$

$az\ X\ a\rho = a\varphi$

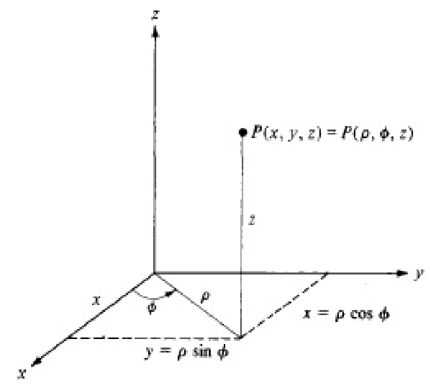


Fig 8: Relation between (x,y,z) to (ρ,φ,z)

Relation between Cartesian and cylindrical coordinates

$\rho = \sqrt{(x^2 + y^2)}$ $\varphi = tan^{-1}\frac{y}{x}$ z = z

Cylindrical to Cartesian coordinates conversion

$X = \rho \cos\varphi$ y =

z= z

Transformation of vector A from Cartesian(A_x , A_y, A_z) to cylindrical (Aρ, Aφ, Az) as

$$\begin{vmatrix} A_\rho \\ A_\varphi \\ A_z \end{vmatrix} = \begin{vmatrix} cos\varphi & sin\varphi & 0 \\ -sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix}$$

The inverse transformation from (Aρ, A φ, Az) to (A_x , A_y, A_z)

$$\begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} = \begin{vmatrix} cos\varphi & -sin\varphi & 0 \\ sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} A_\rho \\ A_\varphi \\ A_z \end{vmatrix}$$

SPHERICAL COORDINATE :

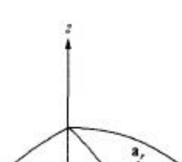


Fig 9 : Point P and unit vectors in spherical co-ordinates

A point P in spherical coordinate can be represented as (r, θ, φ)

r → radius of the sphere , θ→ colatitude angle (angle between z- axis and position vector P)

Φ → azimuth angle measured from the x – axis in xy – plane

The ranges of variables

$0 \leq r \leq \infty$, $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$

A vector A in spherical coordinates may be written as

$$A = A_r\, a_r + A_\theta a_\theta + A_\varphi a_\varphi$$

a_r, a_θ, a_φ → unit vectors in r, θ, φ directions and are orthogonal

The magnitude of A is $|A| = \sqrt{A_r^2 + A_\theta^2 + A_\varphi^2}$

Unit vector relations:

$$a_r \,.\, a_{r=}\, a_\theta \,.\, a_\theta = a_\varphi \,.\, a_\varphi = 1$$

$$a_r \,.\, a_\theta = a_\theta \,.\, a_\varphi = a_\varphi \,.\, a_r = 0$$

$$a_r \, X \, a_\theta = a_\varphi \quad a_\theta \, X \, a_\varphi = a_r \quad a_\varphi \, X \, a_r = a_\theta$$

Cartesian to spherical coordinate conversion :

$$r = \sqrt{(x^2 + y^2 + z^2)} \;, \theta = tan^{-1} \frac{\sqrt{x^2+y^2}}{z} \;, \varphi = tan^{-1}\left(\frac{y}{x}\right)$$

Spherical to cartesian coordinate conversion :

$x = r\, sin\theta \cos\varphi\,, \quad y = r \sin\theta \sin\varphi\,, \quad z = r\cos\varphi$

(A_x , A_y , A_z) → (A_r, A_Θ, A_Φ) vector transformation is performed according to

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix} = \begin{bmatrix} sin\theta \cos\varphi & \sin\theta \sin\varphi & \cos\theta \\ cos\theta \cos\varphi & cos\theta \sin\varphi & -\, sin\theta \\ -sin\varphi & cos\varphi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

(A_r, A_Θ, A_Φ) → (A_x , A_y , A_z) vector transformation is

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} sin\theta \cos\varphi & \cos\theta \cos\varphi & -\sin\varphi \\ sin\theta \sin\varphi & cos\theta \sin\varphi & cos\varphi \\ cos\theta & -sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix}$$

Differential length $dl = dr\, a_r + r\, d\theta\, a_\theta + r \sin\theta\, d\varphi\, a_\varphi$

Differential surface $ds = r^2 sin\theta\, d\theta d\varphi a_r = r\, sin\theta dr\, d\varphi\, a_\theta = r\, dr\, d\theta\, a_\varphi$

Differential volume $dv = r^2 sin\theta\, dr\, d\theta\, d\varphi$

DEL OPERATOR :

The del operator written as ∇ , is the vector differential operator . It is also known as gradient operator.

The operator is useful in defining

- The gradient of a scalar V, ∇V
- The divergence of a vector A, $\nabla . A$
- The curl of a vector A, $\nabla\, X\, A$
- The laplacian of a scalar V, $\nabla^2 V$

The del operator in Cartesian coordinate is

$$\nabla = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z$$

The del operator in Cylindrical coordinate is

$$\nabla = a_\rho \frac{\partial}{\partial \rho} + a_\varphi \frac{1}{\rho} \frac{\partial}{\partial \varphi} + a_z \frac{\partial}{\partial z}$$

The del operator in spherical coordinate is

$$\nabla = a_r \frac{\partial}{\partial r} + a_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + a_\varphi \frac{1}{r\, sin\theta} \frac{\partial}{\partial \varphi}$$

GRADIENT OF A SCALAR:

Gradient of a scalar field V is a vector that represents both the magnitude and the direction of the maximum space rate of increase of V.

Mathematically expressed as $dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$

$$= \left(\frac{\partial V}{\partial x}a_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z\right).\left(d_x a_x + d_y a_y + d_z a_z\right)$$

$$G = \frac{\partial V}{\partial x}a_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z$$

$$dv = G.dl = G\cos\theta\ dl$$

$$\frac{dv}{dl} = G\cos\theta$$

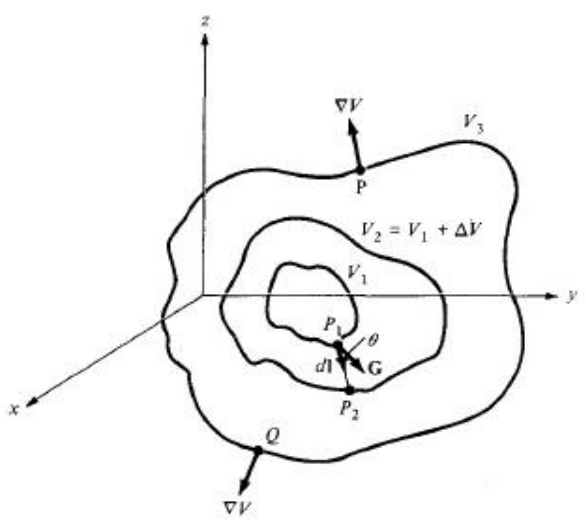


Fig 10. Gradient of a scalar

G → gradient of V

$$\nabla V = \frac{\partial V}{\partial x}a_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z$$

For Cartesian coordinates $\nabla V = \frac{\partial V}{\partial x}a_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z$

For Cylindrical coordinates $\nabla V = \frac{\partial V}{\partial \rho} a_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \varphi} a_\varphi + \frac{\partial V}{\partial z} a_z$

For spherical coordinates , $\nabla V = \frac{\partial V}{\partial r} a_r + \frac{1}{r}\frac{\partial V}{\partial \theta} a_\theta + \frac{1}{r\, sin\theta}\frac{\partial V}{\partial \varphi} a_\varphi$

Computation formula on gradient

1. $\nabla(V + U) = \nabla V + \nabla U$
2. $\nabla(VU) = V\nabla U + U\nabla V$
3. $\nabla \left[\frac{V}{U}\right] = \frac{U\nabla V - V\nabla U}{U^2}$
4. $\nabla V^n = nV^{n-1}\nabla V$

   Where U and V are scalars and n is an integer.

DIVERGENCE OF A VECTOR AND DIVERGENCE THEOREM :

The net outflow of the flux of a vector field A from a closed surface S is obtained from the integral $\oint A\,.\,ds$. Divergence of A is defined as the net outward flow of flux per unit volume over a closed incremental surface.

The divergence of A at a given point P is the outward flux per unit volume as the volume shrinks about P.

$$div\, A = \nabla . A = \lim_{\Delta V \to 0} \frac{\oint A.ds}{\Delta V}$$

Where ΔV → volume enclosed by closed surface S in which P is located.

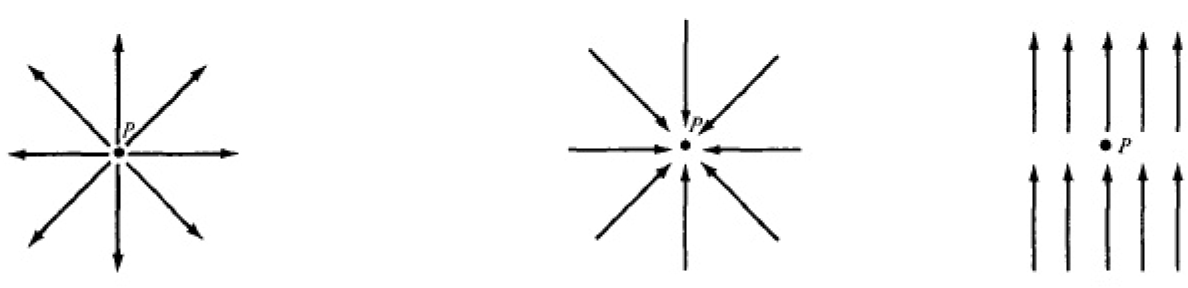


Fig (a) Positive divergence Fig (b) Negative divergenc Fig(c) Zero divergence

Fig 11: Divergence of a vector field at P

Divergence of vector field A at a given point is a measure of how much the field diverges from that point. The divergence of vector field can be viewed as the limit of field source strength per unit volume; it is positive at source point in the field and negative at a sink point or zero where there is neither sink nor source. Evaluate $\nabla . A$ in

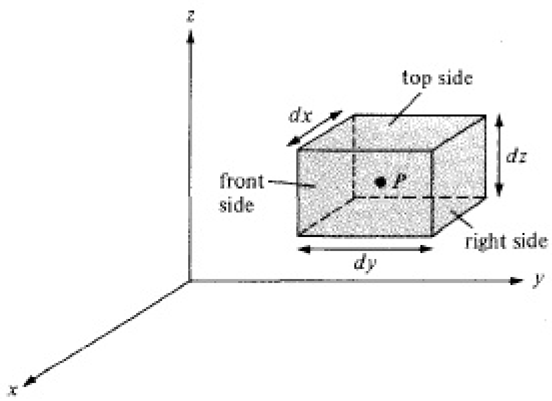


Fig 12 : Evaluation of $\nabla . A$ at point P (x_0,y_0,z_0)

Cartesian coordinate at point P (x_0,y_0,z_0)

$$\oint A.ds = \left(\int_{front} + \int_{back} + \int_{left} + \int_{right} + \int_{top} + \int_{bottom}\right) A.ds \quad \text{--(A)}$$

A three dimensional Taylor series expansion of A_x about P is

$$A_x(x,y,z) = A_x(x_0,y_0,z_0) + (x-x_0)\frac{\partial A_x}{\partial x}\bigg|_P + (y-y_0)\frac{\partial A_y}{\partial y}\bigg|_P + (z-z_0)\frac{\partial A_z}{\partial z}\bigg|_P + \textit{higher order terms}$$

For the front side , $x=x_0 + dx/2$ and $ds = dydz\, a_x$, Then

$$\int_{front} A.ds = dydz\left[A_x(x_0,y_0,z_0) + \frac{dx}{2}\frac{\partial A_x}{\partial x}\bigg|_P\right] + \text{higher order terms}$$

For back side , $x=x_0 - dx/2$ and $ds = dydz\,(-a_x)$, Then

$$\int_{back} A.ds = -\,dydz\left[A_x(x_0,y_0,z_0) - \frac{dx}{2}\frac{\partial A_x}{\partial x}\bigg|_P\right] + \text{higher order terms}$$

Hence

$$\int_{front} A.ds + \int_{back} A.ds = dx\, dydz\, \frac{\partial A_x}{\partial x}\bigg|_P + \text{higher order terms} \quad \text{--(1)}$$

By taking similar steps

$$\int_{left} A.ds + \int_{right} A.ds = dx\, dydz\, \frac{\partial A_y}{\partial y}\bigg|_P + \text{higher order terms} \quad \text{-- (2)}$$

And

$$\int_{top} A.ds + \int_{bottom} A.ds = dx\, dydz\, \frac{\partial A_z}{\partial z}\bigg|_P + \text{higher order terms} \quad \text{--(3)}$$

Substitute equation 1 to 3 in equation A.

$$\lim_{\Delta V \to 0} \oint_s \frac{A.ds}{\Delta v} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)\bigg|_{atP}$$

Because the higher – order terms will vanish as $\Delta v \rightarrow 0$.Thus , the divergence of A at point P (x_0,y_0,z_0) in a Cartesian system is given by

$$\nabla. A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Similarly the divergence in Cylindrical coordinates is

$$\nabla. A = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

The divergence in spherical coordinates is

$$\nabla. A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r sin\theta}\frac{\partial}{\partial \theta}(A_\theta sin\theta) + \frac{1}{r sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$$

Properties of divergence

1.Divergence produces scalar field.

2. $\nabla.(A + B) = \nabla. A + \nabla. B$

3.$\nabla.(VA) = V\nabla. A + A. \nabla V$

The mathematical expression obtained from the definition of divergence is

$$\oint_s A.ds = \int_v \nabla.A\,dv$$

The above expression is called divergence theorem or gauss- Ostrogradsky theorem.

The divergence theorem states that the total outward flux of a vector field A through the closed surface S is same as the volume integral of the divergence of A.

Proof:

Closed Volume V is subdivided into large number of small cells. If K^{th} cell has volume ΔV_k and is bounded by surface S_k

$$\oint_s A.ds = \sum_K \oint_{S_k} A.ds = \sum_K \frac{\oint_{S_k} A\,.ds}{\Delta V_k} \Delta v_k$$

Since the outward flux to one cell is inward to some neighboring cells, there is a cancellation on every interior surface, so the sum of surface integrals over S_k's is same as the surface integrals over the surface S.

$$\oint_s A\,.ds = \int_v \nabla\,.A\,dv$$

The theorem applied to any volume v bounded by closed surface S and $\nabla.A$ is Continuous in the region.

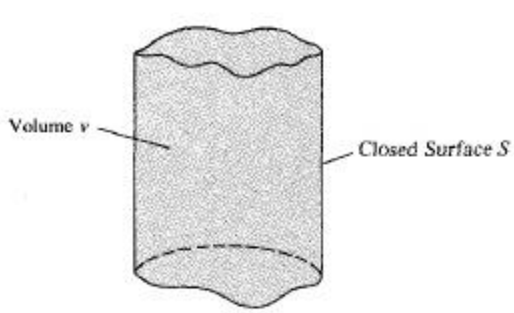


Fig 13: volume V enclosed by surface S

CURL of a Vector and Stokes's theorem:

The circulation of vector field around a closed path L as $\oint_L A \, . \, dl.$

The curl of A is an axial (or rotational) vector whose magnitude is the maximum circulation of A per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.

$$\text{curl A} = \nabla X A = \left(\lim_{\Delta S \to 0} \frac{\oint_L A \, .dl}{\Delta S} \right) a_n \quad \text{---(1)}$$
$$max$$

$\Delta S \rightarrow$ is bounded by curve of length L

a_n → Unit vector normal to the surface S(determined by right handed rule)

The expression for ∇ X A from the definition is

$$\oint_L A.dl = \left(\int_{ab} + \int_{bc} + \int_{cd} + \int_{da} \right) A.dl \quad \text{---(2)}$$

Expand the field component using Taylor series about the centre point $P(x_0,y_0,z_0)$.On side ab ,

$dl = d_y a_y$ and $Z= Z_0 - dz/2$

$$\int_{ab} A \, . \, dl = dy \left[A_y(x_0, y_0, z_0) - \frac{dz}{2} \frac{\partial A_y}{\partial z} \bigg|_P \right] \quad (3)$$

On side bc , $dl = d_z a_z$ and $y = y_0 + dy/2$, so

$$\int_{bc} A \, . \, dl = dz \left[A_z(x_0, y_0, z_0) + \frac{dy}{2} \frac{\partial A_z}{\partial y} \bigg|_P \right] \text{--(4)}$$

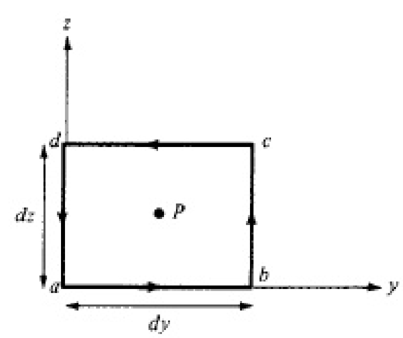


Fig14 : Contour used in evaluating the x component of $\nabla X A$ at point $P(x_0,y_0,z_0)$.

On side cd, $dl = d_y a_y$ and $z = z_0 + dz/2$, so

$$\int_{cd} A \, . \, dl = -dy \left[A_y(x_0, y_0, z_0) + \frac{dz}{2} \frac{\partial A_y}{\partial z} \bigg|_P \right] \text{---(5)}$$

On side da, $dl = d_z a_z$ and $y = y_0 - dy/2$, so

$$\int_{da} A \,.\, dl = -\,dz\left[A_z(x_0, y_0, z_0) - \frac{dy}{2}\frac{\partial A_z}{\partial y}\bigg|_P\right] \quad \text{---(6)}$$

Substitute equation (3) to (6) in equation (2) , $\Delta S = dy\, dz$

$$\lim_{\Delta S \to 0} \oint_L \frac{A.dl}{\Delta S} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

Or

$$(curl\ A)_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \quad \text{---(7)}$$

The y – and x – component of the curl of A can be found by the same way

$$(curl\ A)_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \quad \text{---(8)}$$

$$(curl\ A)_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \quad \text{---(9)}$$

- In Cartesian or rectangular co-ordinate system x, y and z component of the curl A are given by

$$\nabla X A = \begin{vmatrix} ax & ay & az \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ax & Ay & Az \end{vmatrix}$$

$$\nabla X A = \left[\frac{\partial Az}{\partial y} - \frac{\partial Ay}{\partial z}\right] a_x + \left[\frac{\partial Ax}{\partial z} - \frac{\partial Az}{\partial x}\right] a_y + \left[\frac{\partial Ay}{\partial x} - \frac{\partial Ax}{\partial y}\right] a_z$$

In cylindrical co-ordinate

$$\nabla X A = \frac{1}{\rho} \begin{vmatrix} a\rho & \rho a\varphi & az \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix}$$

$$\nabla X\ A = \frac{1}{\rho}\left[\frac{\partial A_z}{\partial \varphi} - \frac{\partial(\rho A_\varphi)}{\partial z}\right] a_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right] a_\varphi + \frac{1}{\rho}\left[\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right] a_z$$

In spherical coordinate

$$\nabla \times A = \frac{1}{r^2 \sin\theta} \begin{vmatrix} a_r & r a_\theta & r \sin\theta \, a_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{vmatrix}$$

$$\nabla X A = \frac{1}{r\, sin\theta}\left[\frac{\partial(A_\varphi sin\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\varphi}\right] a_r + \frac{1}{r}\left[\frac{1}{sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial(r\ A_\varphi)}{\partial r}\right] a_\theta + \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right] a_\varphi$$

Properties of Curl:

1.The curl of vector field is another vector field

2.The curl scalar field V ,$\nabla X V$, make no sense

3.$\nabla X (A + B) = \nabla X A + \nabla X B$

4. $\nabla X(A X B) = A(\nabla . B) - B (\nabla . A) + (B . \nabla A - (A . \nabla)B)$

5. $\nabla X VA = V\nabla XA + \nabla V X A$

6. $\nabla . (\nabla X A) = 0$

7. $\nabla X \nabla V = 0$

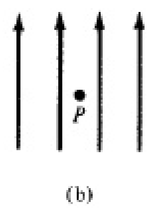


Fig 15 a: Curl at P out of the page

Fig15 b: Curl at P is zero

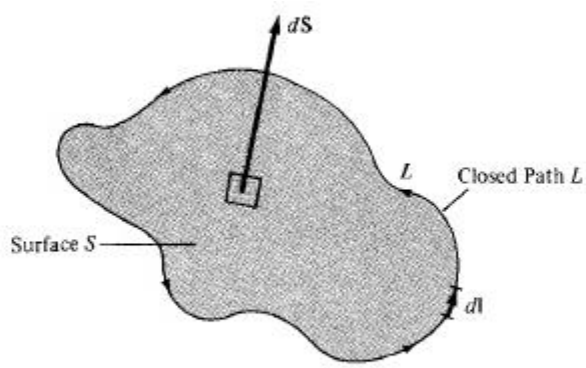


Fig16 : Determining the sense of dl and ds involved in stokes 's theorem.

Definition of curl can be Mathematically expressed as $\oint A.dL = \int_s (\nabla XA).ds$

dL → perimeter of the total surface S.

The above is called Stokes theorem

The stokes theorem states that the circulation of a vector field A around a (closed) path L is equal to the surface integral of the curl of A over the open surface 'S' bounded by L. A and ∇XA are continuous on S.

Proof:

The surface S is divided into a large number of cells . If k[th] cell has a surface area Δ_{Sk} and is bounded by path L_k.

$$\oint_L A.dl = \sum_k \oint_{L_k} A.dl = \sum_k \frac{\oint_{L_k} A.dl}{\Delta S_k} \Delta S_k$$

There is a cancellation of field on every interior path, the sum of line integral around L_k is same as line integral around the bounding curve L.

Taking the limit ΔS_k → 0 , the above equation becomes

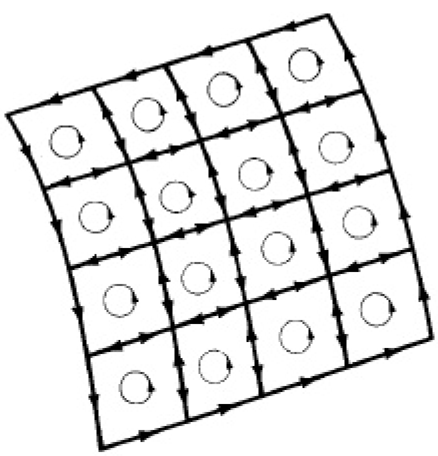


Fig 17: Illustration of stokes theorem

$$\oint A.dL = \int_s (\nabla XA).ds$$

The direction of dl and ds determined by right hand rule.