UNIT- I
COMPLEX DIFFERENTIATION

ANALYTIC FUNCTIONS- Cauchy – Riemann equations

1.1 INTRODUCTION

The theory of functions of a complex variable is the most important in solving a large number of Engineering and Science problems. Many complicated intergrals of real function are solved with the help of a complex variable.

1.1 (a) Complex Variable

$x + iy$ is a complex variable and it is denoted by z.

$(i.e.)z = x + iy \; where \; i = \sqrt{-1}$

1.1 (b) Function of a complex Variable

If $z = x + iy$ and $w = u + iv$ are two complex variables, and if for each value of z in a given region R of complex plane there corresponds one or more values of w is said to be a function z and is denoted by $w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$where $u(x, y)$and $v(x, y)$are real functions of the real variables x and y.

Note:

(i) single-valued function

If for each value of z in R there is correspondingly only one value of w, then w is called a single valued function of z.

Example: $w = z^2, w = \frac{1}{z}$

$w = z^2$					$w = \frac{1}{z}$				
z	1	2	-2	3	z	1	2	-2	3
w	1	4	4	9	w	1	$\frac{1}{2}$	$\frac{1}{-2}$	$\frac{1}{3}$

(ii) Multiple – valued function

If there is more than one value of w corresponding to a given value of z then w is called multiple – valued function.

Example: $w = z^{1/2}$

$w = z^{1/2}$

z	4	9	1
w	$-2,2$	$3,-3$	$1,-1$

(iii) The distance between two points $z$and z_o is $|z-z_o|$

(iv) The circle C of radius δ with centre at the point $z_o$can be represented by $|z-z_o|=\delta$.

(v) $|z-z_o|<\delta$ represents the interior of the circle excluding its circumference.

(vi) $|z-z_o|\leq\delta$ represents the interior of the circle including its circumference.

(vii) $|z-z_o|>\delta$ represents the exterior of the circle.

(viii) A circle of radius 1 with centre at origin can be represented by $|z|=1$

1.1 (c) Neighbourhood of a point z_o

Neighbourhood of a point z_o, we mean a sufficiently small circular region [excluding the points on the boundary] with centre at z_o.

$$(i.e.)\ |z-z_o|<\delta$$

Here, δ is an arbitrary small positive number.

1.1 (d) Limit of a Function

Let $f(z)$ be a single valued function defined at all points in some neighbourhood of point z_o.

Then the limit of $f(z)$ as z approaches $z_o is\ w_o$.

$$(i.e.)\ \lim_{z\to z_o} f(z)=w_o$$

1.1 (e) Continuity

If $f(z)$ is said to continuous at $z=z_o$ then

$$\lim_{z\to z_o} f(z)=f(z_o)$$

If two functions are continuous at a point their sum, difference and product are also continuous at that point, their quotient is also continuous at any such point $[dr\neq 0]$

Example: 1.1 State the basic difference between the limit of a function of a real variable and that of a complex variable. [A.U M/J 2012]

Solution:

In real variable, $x\to x_0$ implies that x approaches x_0 along the X-axis (or) a line parallel to the

X-axis.

In complex variables, $z\to z_0$ implies that z approaches z_0 along any path joining the points $z\ and\ z_0$ that lie in the z-plane.

1.1 (f) Differentiability at a point

A function $f(z)$ is said to be differentiable at a point, $z = z_0$ if the limit

$$f(z_0) = \underset{\Delta z \to 0}{\text{Lt}} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \text{ exists.}$$

This limit is called the derivative of $f(z)$at the point $z = z_0$

If $f(z)$ is differentiable at z_0, then $f(z)$is continuous at z_0. This is the necessary condition for differentiability.

Example: 1.2 If $f(z)$ is differentiable at z_0, then show that it is continuous at that point.

Solution:

As $f(z)$ is differentiable at z_0, both $f(z_0)$ and $f'(z_0)$exist finitely.

$$\text{Now, } \lim_{z \to z_o} |f(z) - f(z_o)| = \lim_{z \to z_o} \frac{f(z) - f(z_o)}{z - z_0}(z - z_0)$$

$$= \lim_{z \to z_o} \frac{f(z) - f(z_o)}{z - z_0} \lim_{z \to z_o}(z - z_0)$$

$$= f'(z_0).0 = 0$$

Hence, $\lim_{z \to z_o} f(z) = \lim_{z \to z_o} f(z_o) = f(z_o)$

As $f(z_0)$ is a constant.

This is exactly the statement of continuity of $f(z)$ at z_0.

Example: 1.3 Give an example to show that continuity of a function at a point does not imply the existence of derivative at that point.

Solution:

Consider the function $w = |z|^2 = x^2 + y^2$

This function is continuous at every point in the plane, being a continuous function of two real variables. However, this is not differentiable at any point other than origin.

Example: 1.4 Show that the function $\mathbf{f(z)}$ is discontinuous at $z = 0$, given that $f(z) = \frac{2xy^2}{x^2+3y^4}$, when $z \neq 0$ and $\mathbf{f(0) = 0}$.

Solution:

Given $f(z) = \frac{2xy^2}{x^2+3y^4}$,

Consider $\lim_{z \to z_o} [f(z)] = \lim_{\substack{y=mx \\ x \to 0}} [f(z)] = \lim_{x \to 0} \frac{2x(mx)^2}{x^2+3(mx)^4} = \lim_{x \to 0} \left[\frac{2m^2x}{1+3m^4x^2}\right] = 0$

$$\lim_{\substack{y^2=x \\ x \to 0}} [f(z)] = \lim_{x \to 0} \frac{2x^2}{x^2 + 3x^2} = \lim_{x \to 0} \frac{2x^2}{4x^2} = \frac{2}{4} = \frac{1}{2} \neq 0$$

$\therefore f(z)$ is discontinuous

Example: 1.5 Show that the function $f(z)$ is discontinuous at the origin $(z = 0)$, given that

$$f(z) = \frac{xy(x-2y)}{x^3+y^3}, \text{ when } z \neq 0$$

$$= 0 \quad , \text{when } z = 0$$

Solution:

$$\text{Consider } \lim_{z \to z_o} [f(z)] = \lim_{\substack{y=mx \\ x\to 0}} [f(z)] = \lim_{x\to 0} \frac{x(mx)(x-2(mx))}{x^3+(mx)^3}$$

$$= \lim_{x\to 0} \frac{m(1-2m)x^3}{(1+m^3)x^3} = \frac{m(1-2m)}{1+m^3}$$

Thus $\lim_{z\to 0} f(z)$ depends on the value of m and hence does not take a unique value.

$\therefore \lim_{z\to 0} f(z)$does not exist.

$\therefore f(z)$ is discontinuous at the origin.

1.1 (A) ANALYTIC FUNCTIONS – NECESSARY AND SUFFICIENT CONDITIONS FOR ANALYTICITY IN CARTESIAN AND POLAR CO-ORDINATES

Analytic [or] Holomorphic [or] Regular function

A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

Entire Function: [Integral function]

A function which is analytic everywhere in the finite plane is called an entire function.

An entire function is analytic everywhere except at $z = \infty$.

Example: $e^z, \sin z, \cos z, sinhz, \cosh z$

1.2 (i) The necessary condition for $f = (z)$ to be analytic. [Cauchy – Riemann Equations]

The necessary conditions for a complex function $f = (\text{z}) = u(x, y) + iv(x, y)$ to be analytic in a region R are $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ $i.e., u_x = v_y \ and \ v_x = -u_y$

[OR]

Derive C – R equations as necessary conditions for a function $\mathbf{w = f(z)}$ to be analytic.

Proof:

Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function at the point z in a region R. Since $f(z)$ is analytic, its derivative $f'(z)$ exists in R

$$f'(z) = \text{Lt}\frac{f(z+\Delta z)-f(z)}{\Delta_z}$$

Let $z = x + iy$

$$\Rightarrow \Delta z = \Delta_x + i\Delta_y$$

$$z + \Delta_z = (x + \Delta_x) + i\left(y + \Delta_y\right)$$

$$f(z) = u(x, y) + iv(x, y)$$

$$f(z + \Delta z) = u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y$$

$$f(z + \Delta z) - f(z) = u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - [u(x, y) + iv(x, y)]$$

$$= [u(x + \Delta x, y + \Delta y) - u(x, y)] + i[v(x + \Delta x, y + \Delta y) - v(x, y)]$$

$$f'(z) = \underset{\Delta z\to 0}{\text{Lt}}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$$= \underset{\Delta z\to 0}{\text{Lt}}\frac{u(x+\Delta x,y+\Delta y)-u(x,y)]+i[v(x+\Delta x,y+\Delta y)-v(x,y)]}{\Delta x+i\Delta y}$$

Case (i)

If $\Delta z \to 0$,firsts we assume that $\Delta y = 0$ and $\Delta x \to 0$.

$$\therefore\ f'(z) = \underset{\Delta x\to 0}{\text{Lt}}\frac{[u(x+\Delta x,y)-u(x,y)]+i[v(x+\Delta x,y)-v(x,y)]}{\Delta x}$$

$$= \underset{\Delta x\to 0}{\text{Lt}}\frac{u(x+\Delta x,y)-u(x,y)}{\Delta x} + \underset{\Delta x\to 0}{\text{Lt}}\frac{v(x+\Delta x,y)-v(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \ldots(1)$$

Case (ii)

If $\Delta z \to 0$ Now, we assume that $\Delta x = 0$ and $\Delta y \to 0$

$$\therefore\ f'(z) = \underset{\Delta y\to 0}{\text{Lt}}\frac{[u(x,y+\Delta y)-u(x,y)]+i[v(x,y+\Delta y)-v(x,y)]}{i\Delta y}$$

$$= \frac{1}{i}\underset{\Delta y\to 0}{\text{Lt}}\frac{u(x,y+\Delta y)-u(x,y)}{\Delta y} + \underset{\Delta y\to 0}{\text{Lt}}\frac{v(x,y+\Delta y)-v(x,y)}{\Delta y}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \ldots(2)$$

From (1) and (2), we get

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -\,i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating the real and imaginary parts we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial v}{\partial y}$$

$$(i.e.)\ u_x = v_y, \quad v_x = -u_y$$

The above equations are known as Cauchy – Riemann equations or C-R equations.

Note: (i) The above conditions are not sufficient for $f(z)$ to be analytic. The sufficient conditions are given in the next theorem.

(ii) Sufficient conditions for $f(z)$ to be analytic.

If the partial derivatives u_x, u_y, v_x and v_y are all continuous in D and $u_x = v_y$ and $u_y = -v_x$, then the function $f(z)$ is analytic in a domain D.

(ii) Polar form of C-R equations

In Cartesian co-ordinates any point z is $z = x + iy$.

In polar co-ordinates, $z = re^{i\theta}$ where r is the modulus and θ is the argument.

Theorem: If $f(z) = u(r,\theta) + iv(r,\theta)$ is differentiable at $z = re^{i\theta}$, then $u_r = \frac{1}{r}v_\theta, v_r = -\frac{1}{r}u_\theta$

(OR) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$

Proof:

Let $z = re^{i\theta}$ and $w = f(z) = u + iv$

$$(i.e.)\, u + iv = f(z) = f(re^{i\theta})$$

Diff. p.w. r. to r, we get

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})\, e^{i\theta} \quad \ldots(1)$$

Diff. p.w. r. to θ, we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\, e^{i\theta} \quad \ldots(2)$$

$$= ri\left[f'(re^{i\theta})\, e^{i\theta}\right]$$

$$= ri\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \text{ by } (1)$$

$$= ri\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}$$

Equating the real and imaginary parts, we get

$$\frac{\partial u}{\partial \theta} = -i\frac{\partial v}{\partial r}, \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

$$(i.e.)\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial v}{\partial \theta}$$

Problems based on Analytic functions – necessary conditions Cauchy – Riemann equations

Example: 1.6 Show that the function $f(z) = xy + iy$ is continuous everywhere but not differentiable anywhere.

Solution:

Given $f(z) = xy + iy$

$(i.e.)$ $u = xy, v = y$

x and y are continuous everywhere and consequently $u(x, y) = xy$ and $v(x, y) = y$ are continuous everywhere.

Thus $f(z)$ is continuous everywhere.

But

$u = xy$	$v = y$
$u_x = y$	$v_x = 0$
$u_y = x$	$v_y = 1$
$u_x \neq v_y$	$u_y \neq -v_x$

C−R equations are not satisfied.

Hence, $f(z)$ is not differentiable anywhere though it is continuous everywhere .

Example: 1.7 Show that the function $f(z) = \overline{z}$ is nowhere differentiable.

Solution:

Given $f(z) = \overline{z} = x - iy$

i.e.,

$u = x$	$v = -y$
$\frac{\partial u}{\partial x} = 1$	$\frac{\partial v}{\partial x} = 0$
$\frac{\partial u}{\partial y} = 0$	$\frac{\partial v}{\partial y} = -1$

$\therefore u_x \neq v_y$

C−R equations are not satisfied anywhere.

Hence, $f(z) = \overline{z}$ is not differentiable anywhere (or) nowhere differentiable.

Example: 1.8 Show that $f(z) = |z|^2$ is differentiable at $z = 0$ but not analytic at $z = 0$.

Solution:

Let $z = x + iy$

$\overline{z} = x - iy$

$|z|^2 = z\,\overline{z} = x^2 + y^2$

$(i.e.)\ f(z) = |z|^2 = (x^2 + y^2) + i0$

$u = x^2 + y^2$	v= 0
$u_x = 2x$ $u_y = 2y$	$v_x = 0$ $v_y = 0$

So, the C−R equations $u_x = v_y$ and $u_y = -v_x$ are not satisfied everywhere except at $z = 0$.
So, f(z) may be differentiable only at $z = 0$.
Now, $u_x = 2x$, $u_y = 2y$,$v_x = 0$ and $v_y = 0$ are continuous everywhere and in particular at (0,0).
Hence, the sufficient conditions for differentiability are satisfied by $f(z)$ at $z = 0$.
So, $f(z)$ is differentiable at $z = 0$ only and is not analytic there.

Inverse function

Let $w = f(z)$ be a function of z and $z = F(w)$ be its inverse function.

Then the function $w = f(z)$ will cease to be analytic at $\frac{dz}{dw} = 0$ and $z = F(w)$ will be so, at point where $\frac{dw}{dz} = 0$.

Example: 1.9 Show that $f(z) = log\ z$ analytic everywhere except at the origin and find its derivatives.

Solution:

Let $z = re^{i\theta}$

$$f(z) = log\ z$$

$$= \log(re^{i\theta}) = logr + \log(e^{i\theta}) = logr + i\theta$$

But, at the origin, $r = 0$. Thus, at the origin,

$$f(z) = log0 + i\theta = -\infty + i\theta$$

So, $f(z)$is not defined at the origin and hence is not differentiable there.

Note : $e^{-\infty} = 0$ $\log e^{-\infty} = \log 0\,; -\infty = log0$

At points other than the origin, we have

$u(r,\theta) = log\ r$	$v(r,\theta) = \theta$
$u_r = \frac{1}{r}$ $u_\theta = 0$	$v_r = 0$ $v_\theta = 1$

So, $logz$ satisfies the C−R equations.

Further $\frac{1}{r}$ is not continuous at $z = 0$.

So, $u_r, u_\theta, v_r, v_\theta$ are continuous everywhere except at $z = 0$. Thus log z satisfies all the sufficient conditions for the existence of the derivative except at the origin. The derivative is

$$f'(z) = \frac{u_r + iv_r}{e^{i\theta}} = \frac{\left(\frac{1}{r}\right)+i(0)}{e^{i\theta}} = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

Note: $f(z) = u + iv \Rightarrow f(re^{i\theta}) = u + iv$

Differentiate w.r.to 'r', we get

$$(i.e.)\ e^{i\theta} f'(re^{i\theta}) = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

Example: 1.10 Check whether $w = \bar{z}$ is analytics everywhere.

Solution:

Let $w = f(z) = \bar{z}$

$u + iv = x - iy$

$u = x$	$v = -y$
$u_x = 1$ $u_y = 0$	$v_x = 0$ $v_y = -1$

$u_x \neq v_y$ at any point p(x,y)

Hence, C−R equations are not satisfied.

∴The function $f(z)$ is nowhere analytic.

Example: 1.11 Test the analyticity of the function $w = \sin z$.

Solution:

Let $w = f(z) = sinz$

$u + iv = \sin(x + iy)$

$u + iv = \sin x \cos iy + \cos x \sin iy$

$u + iv = \sin x \cosh y + i \cos x \sin hy$

Equating real and imaginary parts, we get

$u = \sin x \cosh y$	$v = \cos x \sinh y$
$u_x = \cos x \cosh y$	$v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$	$v_y = \cos x \cosh y$

$\therefore u_x = v_y$ and $u_y = -v_x$

C −R equations are satisfied.

Also the four partial derivatives are continuous.

Hence, the function is analytic.

Example: 1.12 Determine whether the function $2xy + i(x^2 - y^2)$ is analytic or not.

Solution:

Let $f(z) = 2xy + i(x^2 - y^2)$

$(i.e.)$

$u = 2xy$	$v = x^2 - y^2$
$\frac{\partial u}{\partial x} = 2y$ $\frac{\partial u}{\partial y} = 2x$	$\frac{\partial v}{\partial x} = 2x$ $\frac{\partial v}{\partial y} = -2y$

$u_x \neq v_y$ and $u_y \neq -v_x$

C−R equations are not satisfied.

Hence, $f(z)$ is not an analytic function.

Example: 1.13 Prove that $f(z) = \cosh z$ is an analytic function and find its derivative.

Solution:

Given $f(z) = \cosh z = \cos(iz) = \cos[i(x + iy]$

$= \cos(ix - y) = \cos ix \cos y + \sin(ix) \sin y$

$u + iv = \cosh x \cos y + i \sinh x \sin y$

$u = \cosh x \cos y$	$v = \sinh x \sin y$
u_x =sinh x cos y $u_y = -\cosh x \sin y$	v_x =cosh x sin y v_y =sinh x cos y

$\therefore u_x, u_y, v_x$ and v_y exist and are continuous.

$u_x = v_y$ and $u_y = -v_x$

C−R equations are satisfied.

$\therefore f(z)$ is analytic everywhere.

Now, $f'(z) = u_x + iv_x$

$= \sinh x \cos y + i \cosh x \sin y$

$= \sinh(x + iy) = \sinh z$

Example: 1.14 If $w = f(z)$ is analytic, prove that $\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$ where $z = x + iy$, and prove that $\frac{\partial^2 w}{\partial z \partial \bar{z}} = 0$.

Solution:

Let $w = u(x, y) + iv(x, y)$

As $f(z)$ is analytic, we have $u_x = v_y, u_y = -v_x$

Now, $\frac{dw}{dz} = f'(z) = u_x + iv_x = v_y - iu_y = i(u_y + iv_y)$

$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]$

$= \frac{\partial}{\partial x}(u + iv) = -i\frac{\partial}{\partial y}(u + iv)$

$= \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$

We know that, $\frac{\partial w}{\partial z} = 0$

$\therefore \frac{\partial^2 w}{\partial z \partial \bar{z}} = 0$

Also $\frac{\partial^2 w}{\partial \bar{z} \partial z} = 0$

Example: 1.15 Prove that every analytic function $w = u(x, y) + iv(x, y)$ can be expressed as a function of z alone.

Proof:

Let $z = x + iy$ and $\bar{z} = x - iy$

$x = \frac{z+\bar{z}}{2}$ and $y = \frac{z+\bar{z}}{2i}$

Hence, u and v and also w may be considered as a function of z and $\bar{z}$

Consider $\frac{\partial w}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}}$

$= \left(\frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}}\right) + \left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}\right)$

$= \left(\frac{1}{2}u_x - \frac{1}{2i}u_y\right) + i\left(\frac{1}{2}v_x - \frac{1}{2i}v_y\right)$

$= \frac{1}{2}(u_x - v_y) + \frac{i}{2}(u_y + v_x)$

$= 0$ by C−R equations as w is analytic.

This means that w is independent of $\bar{z}$

$(i.e.)$ w is a function of z alone.

This means that if $w = u(x,y) + iv(x,y)$ is analytic, it can be rewritten as a function of $(x + iy)$.

Equivalently a function of $\overline{z}$ cannot be an analytic function of z.

Example: 1.16 Find the constants a, b, c if $f(z) = (x + ay) + i(bx + cy)$ is analytic.

Solution:

$$f(z) = u(x,y) + iv(x,y)$$

$$= (x + ay) + i(bx + cy)$$

$u = x + ay$	$v = bx + cy$
$u_x = 1$ $u_y = a$	$v_x = b$ $v_y = c$

Given $f(z)$ is analytic

$\Rightarrow u_x = v_y$ and $u_y = -v_x$

$1 = c$ and $a = -b$

Example: 1.17 Examine whether the following function is analytic or not $f(z) = e^{-x}$(cos y – i sin y).

Solution:

Given $f(z) = e^{-x}(\cos y - i \sin y)$

$\Rightarrow u + iv = e^{-x} \cos y - ie^{-x} \sin y$

$u = e^{-x} \cos y$	$v = -e^{-x} \sin y$
$u_x = -e^{-x} \cos y$ $u_y = -e^{-x} \sin y$	$v_x = e^{-x} \sin y$ $v_y = -e^{-x} \cos y$

Here, $u_x = v_y$ and $u_y = -v_x$

$\Rightarrow$ C−R equations are satisfied

$\Rightarrow f(z)$ is analytic.

Example: 1.18 Test whether the function $f(z) = \frac{1}{2}\log(x^2 + y^2 + \tan^{-1}\left(\frac{y}{x}\right)$ is analytic or not.

Solution:

Given $f(z) = \frac{1}{2}\log(x^2 + y^2 + i \tan^{-1}\left(\frac{y}{x}\right)$

$(i.e.) u + iv = \frac{1}{2}\log(x^2 + y^2 + i \tan^{-1}\left(\frac{y}{x}\right)$

$u = \frac{1}{2}\log(x^2+y^2)$	$v = \tan^{-1}\left(\frac{y}{x}\right)$
$u_x = \frac{1}{2}\frac{1}{x^2+y^2}(2x)$ $= \frac{x}{x^2+y^2}$ $u_y = \frac{1}{2}\frac{1}{x^2+y^2}(2y)$ $= \frac{y}{x^2+y^2}$	$v_x = \frac{1}{1+\frac{y^2}{x^2}}\left[-\frac{y}{x^2}\right]$ $= \frac{-y}{x^2+y^2}$ $v_y = \frac{1}{1+\frac{y^2}{x^2}}\left[\frac{1}{x}\right]$ $= \frac{x}{x^2+y^2}$

Here, $u_x = v_y$ and $u_y = -v_x$

$\Rightarrow$ C−R equations are satisfied

$\Rightarrow f(z)$ is analytic.

Example: 1.19 Find where each of the following functions ceases to be analytic.

(i) $\frac{z}{(z^2-1)}$ **(ii)** $\frac{z+i}{(z-i)^2}$

Solution:

(i) Let $f(z) = \frac{z}{(z^2-1)}$

$$f'(z) = \frac{(z^2-1)(1)-z(2z)}{(z^2-1)^2} = \frac{-(z^2+1)}{(z^2-1)^2}$$

$f(z)$ is not analytic, where $f'(z)$ does not exist.

$(i.e.)\ f'(z) \to \infty$

$(i.e.)(z^2-1)^2 = 0$

$(i.e.)\quad z^2 - 1 = 0$

$z = 1$

$z = \pm 1$

$\therefore f(z)$ is not analytic at the points $z = \pm 1$

(ii) Let $f(z) = \frac{z+i}{(z-i)^2}$

$$f'(z) = \frac{(z-i)^2(1)(z+i)[2(z-i)]}{(z-i)^4} = \frac{(z+3i)}{(z-i)^3}$$

$f'(z) \to \infty, at\ z = i$

$\therefore f(z)$ is not analytic at $z = i$.