1.3 CONSTRUCTION OF ANALYTIC FUNCTION

There are three methods to find $f(z)$.

Method: 1 Exact differential method.

(i) Suppose the harmonic function $u(x, y)$ is given.

Now, $dv = v_x dx + v_y dy$ is an exact differential

Where, v_x and v_y are known from u by using C−R equations.

$\therefore v = \int v_x\, dx + \int v_y dy = -\int u_y dx + \int u_x dy$

(ii) Suppose the harmonic function $v(x, y)$ is given.

Now, $du = u_x dx + u_y dy$ is an exact differential

Where, u_x and u_y are known from v by using C−R equations.

$$u = \int u_x\, dx + \int u dy$$
$$= \int v_y dx + \int -v_x dy$$
$$= \int v_y dx - \int v_x dy$$

Method: 2 Substitution method

$$f(z) = 2u\left[\frac{1}{2}(z+a), \frac{-i}{2}(z-a)\right] - [u(a,0), -iv(a,0)]$$

Here, $u(a, 0), -iv(a, 0)$ is a constant

Thus $f(z) = 2u\left[\frac{1}{2}(z+a), \frac{-i}{2}(z-a)\right] + C$

By taking $a = 0$, that is, if $f(z)$ is analytic $z = 0 + i0$,

We have the simpler formula for $f(z)$

$$f(z) = 2\left[u\frac{z}{2}, \frac{-iz}{2}\right] + C$$

Method: 3 [Milne – Thomson method]

(i) To find $f(z)$ when u is given

Let $f(z) = u + iv$

$$f'(z) = u_x + iv_x$$
$$= u_x - iv_y \text{ [by C−R condition]}$$

$\therefore f(z) = \int u_x\,(z, 0)dz - i\int u_y(z, 0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

(ii) To find $f(z)$ when v is given

Let $f(z) = u + iv$

$$f'(z) = u_x + iv_x$$

$= v_y + iv_x$ [by C−R condition]

$\therefore f(z) = \int v_y\,(z,0)dz + i\int v_x(z,0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

Example: 1.22 Construct the analytic function $f(z)$ for which the real part is $e^x \cos y$.

Solution:

Given $u = e^x \cos y$

$\Rightarrow u_x = e^x \cos y$ [$\therefore \cos 0 = 1$]

$\Rightarrow u_x(z,0) = e^x$

$\Rightarrow u_y = e^x \cos y$ [$\therefore \sin 0 = 0$]

$\Rightarrow u_y(z,0) = 0$

$\therefore f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

$\therefore f(z) = \int e^z\,dz - i\int 0\,dz + C$

$= e^z + C$

Example: 1.23 Determine the analytic function $w = u + iv$ if $u = e^{2x}(x\cos 2y - y\sin 2y)$

Solution:

Given $u = e^{2x}(x\cos 2y - y\sin 2y)$

$u_x = e^{2x}[\cos 2y] + (xcos2y - y\sin 2y)[2\,e^{2x}]$

$u_x(z,0) = e^{2z}[1]+[z(1) - 0][2e^{2z}]$

$= e^{2z} + 2ze^{2z}$

$= (1+2z)e^{2x}$

$u_y = e^{2x}[-2x\sin 2y - (y2cos2y + \sin 2y)]$

$u_y(z,0) = e^{2z}[-0 - (0+0)] = 0$

$\therefore f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

$f(z) = \int(1+2z)e^{2z}dz - i\int 0 + dz + C$

$= \int(1+2z)e^{2z}dz + C$

$= (1+2z)\frac{e^{2z}}{2} - 2\frac{e^{2z}}{4} + C$ [$\because \int uv\,dz = uv_1 - u'v_2 + u''v_3 - \cdots$]

$= \frac{e^{2z}}{2} + ze^{2z} - \frac{e^{2z}}{2} + C$

$= ze^{2z} + C$

Example: 1.24 Determine the analytic function where real part is

$$\boldsymbol{u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1}.$$

Solution:

Given $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$u_x = 3x^2 - 3y^2 + 6x$

$\Rightarrow u_x(z, 0) = 3z^2 - 0 + 6z$

$u_y = 0 - 6xy + 0 - 6y$

$\Rightarrow u_y(z, 0) = 0$

$f(z) = \int u_x\,(z, 0)dz - i \int u_y(z, 0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

$f(z) = \int (3z^2 + 6z)dz - i \int 0 + dz + C$

$= 3\frac{z^2}{3} + 6\frac{z^2}{2} + C$

$= z^3 + 3z^2 + C$

Example: 1.25 Determine the analytic function whose real part in $\frac{\sin 2x}{\cosh 2y - \cos 2x}$

Solution:

Given $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$u_x = \frac{(\cosh 2y - \cos 2x)[2\cos 2x] - \sin 2x[2\sin 2x]}{[\cosh 2y - \cos 2x]^2}$

$u_x(z, 0) = \frac{(1 - \cos 2z)(2\cos 2z) - 2\sin^2 2z}{[\cosh 0 - \cos 2z]^2}$

$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$

$= \frac{2\cos 2z - 2[\cos^2 2z + \sin^2 2z]}{(1 - \cos 2z)^2}$

$= \frac{2\cos 2z - 2}{(1 - \cos 2z)^2}$

$= \frac{-2(1 - \cos 2z)}{(1 - \cos 2z)^2}$

$= \frac{2\cos 2z - 2}{(1 - \cos 2z)}$

$= \frac{-2}{2sin^2 2}$

$= -cosec^2 z$

$u_y = \frac{(\cosh 2y - \cos 2x)\,(0) - \sin 2x[2\sin 2y]}{[\cosh 2y - \cos 2x]^2}$

$\Rightarrow u_y(z, 0) = 0$

$$f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C$$ [by Milne−Thomson rule],

where C is a complex constant.

$$f(z) = \int(-cosec^2 z)dz - i\int 0\,dz + C$$
$$= cot\,z + C$$

Example: 1.26 Show that the function $u = \frac{1}{2}\log(x^2 + y^2)$ is harmonic and determine its conjugate. Also find $f(z)$

Solution:

Given $u = \frac{1}{2}\log(x^2 + y^2)$

$$u_x = \frac{1}{2}\frac{1}{(x^2+y^2)}(2x) = \frac{x}{x^2+y^2},$$
$$\Rightarrow u_x(z,0) = \frac{z}{z^2} = \frac{1}{z}$$
$$u_{xx} = \frac{(x^2+y^2)[1]-x[2x]}{[x^2+y^2]^2} = \frac{x^2+y^2-2x^2}{[x^2+y^2]^2} = \frac{y^2-x^2}{[x^2+y^2]^2} \quad \ldots(1)$$
$$u_y = \frac{1}{2}\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2}$$
$$\Rightarrow u_y(z,0) = 0$$
$$u_{yy} = \frac{(x^2+y^2)[1]-y[2y]}{[x^2+y^2]^2} = \frac{x^2-y^2}{[x^2+y^2]^2} \quad \ldots(2)$$

To prove u is harmonic:

$$\therefore u_{xx} + u_{yy} = \frac{(y^2-x^2)+(x^2-y^2)}{[x^2+y^2]^2} = 0 \qquad by\ (1)\&(2)$$

$\Rightarrow u$ is harmonic.

To find $f(z)$:

$$f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C$$ [by Milne−Thomson rule],

Where, C is a complex constant.

$$f(z) = \int \frac{1}{z}dz - i\int 0\,dz + C$$
$$= log\,z + C$$

To find v :

$$f(z) = log\,(re^{i\theta}) \qquad [\because z = re^{i\theta}]$$
$$u + iv = \log r + \log\, e^{i\theta} = \log r + i\theta$$
$$\Rightarrow u = logr, v = \theta$$

Note: $z = x + iy$

$$r = |z| = \sqrt{x^2 + y^2}$$
$$\log r = \frac{1}{2}\log(x^2 + y^2)$$

$\tan\theta = \frac{y}{x}$

$\theta = tan^{-1}\left(\frac{y}{x}\right) \qquad i.e., v = tan^{-1}\left(\frac{y}{x}\right)$

Example: 1.27 Construct an analytic function $f(z) = u + iv$, given that $u = e^{x^2-y^2}\cos 2xy$. Hence find v. **[A.U D15/J16, R-08]**

Solution:

Given $u = e^{x^2-y^2}\cos 2xy = e^{x^2}e^{-y^2}\cos 2xy$

$$u_x = e^{-y^2}\left[e^{x^2}(-2y\sin 2xy) + \cos 2xy\ e^{x^2}2x\right]$$

$$u_x(z,0) = 1\left[e^{z^2}(0) + 2ze^{z^2}\right] = 2ze^{z^2}$$

$$u_y = e^{x^2}\left[e^{-y^2}(-2x\sin 2xy) + \cos 2xy e^{-y^2}(-2y)\right]$$

$$u_y(z,0) = e^{z^2}[0+0] = 0$$

$f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule]

$$= \int 2z\,e^{z^2}dz + C$$

$$= 2\int z\,e^{z^2}dz + C$$

put $t = z^2$, dt= $2z\,dz$

$$= \int e^t\,dt + C$$

$$= e^t + C$$

$$f(z) = e^{z^2} + C$$

To find v :

$$u + iv = e^{(x+iy)^2} = e^{x^2-y^2+i\,2xy} = e^{x^2-y^2}e^{i2\,xy}$$

$$= e^{x^2-y^2}\,[\cos(2xy) + isin\,(2xy)]$$

$v = e^{x^2-y^2}\sin 2xy$ [∵equating the imaginary parts]

Example: 1.28 Find the regular function whose imaginary part is $e^{-x}(x\cos y + y\sin y)$.

Solution:

Given $v = e^{-x}(x\cos y + y\sin y)$

$$v_x = e^{-x}[\cos y] + (x\cos y + y\sin y)[-e^{-x}]$$

$$v_x(z,0) = e^{-z} + (z)(-e^{-z}) = (1-z)e^{-z}$$

$$v_y = e^{-x}[-x\sin y + (y\cos y + \sin y\,(1))]$$

$$v_x(z,0) = e^{-z}[0+0+0] = 0$$

$\therefore f(z) = \int v_y(z,0)dz + i\int v_x(z,0)dz + C$ [by Milne−Thomson rule]

Where, C is a complex constant.

$$f(z) = \int 0dz + i\int (1-z)e^{-z}dz + C$$
$$= i\int (1-z)e^{-z}dz + C$$
$$= i\left[(1-z)\left[\frac{e^{-z}}{-1}\right] - (-1)\left[\frac{e^{-z}}{(-1)^2}\right]\right] + C$$
$$= i[-(1-z)e^{-z} + e^{-z}] + C$$
$$= ize^{-z} + C$$

Example: 1.29 In a two dimensional flow, the stream function is $\psi = tan^{-1}\left(\frac{y}{x}\right)$. Find the velocity potential $\boldsymbol{\varphi}$.

Solution:

Given $\psi = tan^{-1}(y/x)$

We should denote, ϕ by u and ψ by v

$$\therefore v = tan^{-1}(y/x)$$

$$v_x = \frac{1}{1+(y/x)^2}\left[\frac{-y}{x^2}\right] = \frac{-y}{x^2+y^2}, \qquad v_x(z,0) = 0$$

$$v_y = \frac{1}{1+(y/x)^2}\left[\frac{1}{x}\right] = \frac{x}{x^2+y^2} \qquad v_x(z,0) = \frac{z}{z^2} = \frac{1}{z}$$

$$\therefore f(z) = \int v_y\,(z,0)dz + i\int v_x(z,0)dz + C$$

$$f(z) = \int \frac{1}{z}dz + i\int 0\,dz + C = \log z + C$$

To find $\boldsymbol{\varphi}$:

$$f(z) = log\,(re^{i\theta}) \qquad [\because z = re^{i\theta}]$$

$$u + iv = \log r + \log\, e^{i\theta}$$

$$u + iv = \log r + i\theta$$

$$\Rightarrow u = \log r$$

$z = x + iy, \lvert z\rvert = \sqrt{x^2+y^2}$

$$\Rightarrow u = log\sqrt{x^2+y^2}$$

$$= \frac{1}{2}\log(x^2+y^2)$$

So, the velocity potential φ is

$$\varphi = \frac{1}{2}\log(x^2+y^2)$$

Note: In two dimensional steady state flows, the complex potential $f(z) = \varphi(x,y) + i\psi(x,y)$ is analytic.

Example: 1.30 If $\boldsymbol{w = u + iv}$ is an analytic function and $\boldsymbol{v = x^2 - y^2 + \frac{x}{x^2+y^2}}$, find $\boldsymbol{u}$.

Solution:

Given $v = x^2 - y^2 + \frac{x}{x^2+y^2}$

$$v_x = 2x - 0 + \frac{(x^2+y^2)(1)-x(2x)}{(x^2+y^2)^2}$$

$$= 2x + \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad v_x(z,0) = 2z + \frac{(-z^2)}{(z^2)}$$

$$\Rightarrow v_x(z,0) = 2z - \frac{1}{z^2}$$

$$v_y = 0 - 2y + \frac{0-x(2y)}{(x^2+y^2)^2}$$

$$= 0 - 2y - \frac{2xy}{(x^2+y^2)^2}$$

$$\Rightarrow \ v_y(z,0) = 0$$

$\therefore f(z) = \int v_y\,(z,0)dz + i\int v_x(z,0)dz + C$ [by Milne−Thomson rule]

Where, C is a complex constant.

$$f(z) = \int 0dz + i\int\left(2z - \frac{1}{z^2}\right)dz + C$$

$$= i\left[2\frac{z^2}{2} + \frac{1}{z}\right] + C \qquad \left[\because \int \frac{-1}{z^2}dz = \frac{1}{z}\right]$$

$$= i\left[z^2 + \frac{1}{z}\right] + C$$

Example: 1.31 If $f(z) = u + iv$ is an analytic function and $u - v = e^x(\cos y - \sin y)$, find $f(z)$ interms of z.

Solution:

Given $u - v = e^x(\cos y - \sin y)$, $\quad \ldots(A)$

Differentiate (A) p.w.r. to *x*, we get

$$u_x - v_x = e^x(\cos y - \sin y),$$

$$u_x(z,0) - v_x(z,0) = e^z \qquad \ldots(1)$$

Differentiate (A) p.w.r. to y, we get

$$u_y - v_y = e^x(-\sin y - \cos y)$$

$$u_y(z,0) - v_y(z,0) = e^z[-1]$$

i.e., $u_y(z,0) - v_y(z,0) = -e^z$

$-v_x(z,0) - u_x(z,0) = -e^z \qquad \ldots(2)$ [by C−R conditions]

$$(1) + (2) \Rightarrow -2v_x(z,0) = 0$$

$$\Rightarrow v_x(z,0) = 0$$

$$(1) \Rightarrow u_x(z,0) = e^z$$

$f(z) = \int u_x\,(z,0)dz + i\int v_x(z,0)dz + C$ [by Milne−Thomson rule]

$$f(z) = \int e^z dz + i0 + C$$

$$= e^z + C$$

Example: 1.32 Find the analytic functions $f(z) = u + iv$ given that

(i) $2u + v = e^x(\cos y - \sin y)$

(ii) $u - 2v = e^x(\cos y - \sin y)$

Solution:

Given (i) $2u + v = e^x(\cos y - \sin y)$ $\quad \ldots(A)$

Differentiate (A) p.w.r. to x, we get

$$2u_x + v_x = e^x(\cos y - \sin y)$$

$$2u_x - u_y = e^x(\cos y - \sin y) \quad \text{[by C−R condition]}$$

$$2u_x(z,0) - u_y(z,0) = e^z \quad \ldots(1)$$

Differentiate (A) p.w.r. to y, we get

$$2u_y + v_y = e^x[-\sin y - \cos y]$$

$$2u_y + u_x = e^x[-\sin y - \cos y] \quad \text{[by C−R condition]}$$

$$2u_y(z,0) + u_x(z,0) = e^z(-1) = -e^z \quad \ldots(2)$$

$$(1) \times (2) \quad \Rightarrow 4u_x(z,0) - 2u_y(z,0) = 2e^z \quad \ldots(3)$$

$$(2) + (3) \quad \Rightarrow 5u_x(z,0) = e^z$$

$$\Rightarrow u_x(z,0) = \frac{1}{5}e^z$$

$$(1) \quad \Rightarrow u_y(z,0) = \frac{2}{5}e^z - e^z = -\frac{3}{5}e^z$$

$$\Rightarrow u_y(z,0) = -\frac{3}{5}e^z$$

$$f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C \quad \text{[by Milne−Thomson rule]}$$

Where, C is a complex constant.

$$f(z) = \int \frac{1}{5}e^z dz - i\int -\frac{3}{5}e^z dz + C$$

$$= \frac{2}{5}e^z + \frac{3}{5}ie^z + C$$

$$= \frac{1+3i}{5}e^z + C$$

(ii) $u - 2v = e^x(\cos y - \sin y)$ $\quad \ldots(B)$

Differrentiate (B) p.w.r. to x, we get

$$u_x - 2v_x = e^x(\cos y - \sin y)$$

$$u_x + 2u_y = e^x(\cos y - \sin y) \quad \text{[by C−R condition]}$$

$$u_x(z,0) + 2u_y(z,0) = e^z \quad \ldots(1)$$

Differentiate (B) p.w.r. to y, we get

$$u_y - 2v_y = e^x[-\sin y - \cos y]$$

$$u_y - 2u_x = e^x\,[-\sin y - \cos y] \quad \text{[by C−R condition]}$$

$$u_y(z,0) - 2u_x(z,0) = -e^z \qquad \ldots(2)$$

$(1) \times (2) \quad \Rightarrow 2u_x(z,0) + 4u_y(z,0) = 2e^z \qquad \ldots(3)$

$(2) + (3) \quad \Rightarrow \; 5u_y(z,0) = e^z$

$$\Rightarrow u_y(z,0) = \frac{1}{5}e^z$$

$(1) \quad \Rightarrow u_x(z,0) = -\frac{2}{5}e^z + e^z$

$$= \frac{3}{5}e^z$$

$$f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C \quad \text{[by Milne−Thomson rule]}$$

Where, C is a complex constant.

$$f(z) = \int \frac{3}{5}e^z dz - i\int \frac{1}{5}e^z dz + C$$

$$= \frac{3}{5}e^z - i\frac{1}{5}e^z + C = \frac{3-i}{5}e^z + C$$

Example: 3.33 Determine the analytic function $f(z) = u + iv$ given that $3u + 2v = y^2 - x^2 + 16xy$

Solution:

Given $3u + 2v = y^2 - x^2 + 16xy \qquad \ldots(A)$

Differentiate (A) p.w.r. to x, we get

$$3u_x + 2v_x = -2x + 16y$$

$$3u_x - 2u_y = -2x + 16y \quad \text{[by C−R condition]}$$

$$3u_x(z,0) - 2u_y(z,0) = -2z \qquad \ldots(1)$$

Differentiate (A) p.w.r. to y, we get

$$3u_y + 2v_y = 2y + 16x$$

$$3u_y + 2u_x = 2y + 16x \quad \text{[by C−R condition]}$$

$$3u_y(z,0) + 2u_x(z,0) = 16z \qquad \ldots(2)$$

$(1) \times (2) \quad \Rightarrow 6u_x(z,0) - 4u_y(z,0) = -4z \qquad \ldots(3)$

$(2) \times (3) \quad \Rightarrow \; 9u_y(z,0) + 6u_x(z,0) = 48z$

$(3) - (4) \quad \Rightarrow -13u_y(z,0) = -52z$

$$\Rightarrow u_y(z,0) = 4z$$

$(1) \quad \Rightarrow 3u_x(z,0) = 8z - 2z = 6z$

$$\Rightarrow u_x(z,0) = 2z$$

$$f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C \quad \text{[by Milne−Thomson rule]}$$

where C is a complex constant.

$$f(z) = \int 2zdz - i\int 4zdz + C$$

$$= 2\,\frac{z^2}{2} - i\frac{4z^2}{2} + C$$

$$= z^2 - i2z^2 + C$$

$$= (1 - i2)z^2 + C$$

Example:3.34 Find an analytic function $f(z) = u + iv$ given that

$$\mathbf{2u + 3v = \frac{\sin 2x}{\cosh 2y - \cos 2x}}$$

Solution:

Given $2u + 3v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

Differentiate p.w.r. to x, we get

$$2u_x + 3v_x = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$2u_x - 3u_y = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2} \quad \text{[by C−R condition]}$$

$$2u_x(z,0) - 3u_y(z,0) = \frac{2\cos 2z(1-\cos 2z) - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2cos^2 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1-\cos 2z)^2} = \frac{-2}{1-\cos 2z}$$

$$= \frac{-2}{2\sin^2 z} = -cosec^2 z$$

$$2u_x(z,0) - 3u_y(z,0) = -cosec^2 z \qquad \dots(1)$$

Differentiate p.w.r. to y, we get

$$2u_y + 3v_y = \frac{0 - \sin 2x(\sinh 2y)}{(\cosh 2y - \cos 2x)^2}(2)$$

$$2u_y + 3u_x = \frac{0 - \sin 2x(\sinh 2y)}{(\cosh 2y - \cos 2x)^2}(2) \quad \text{[by C − R condition]}$$

$$2u_y(z,0) + 3u_x(z,0) = 0 \qquad \dots(2)$$

Solving (1) & (2) we get,

$$\Rightarrow u_x(z,0) = -\frac{2}{13}\text{cosec}^2 z$$

$$\Rightarrow u_y(z,0) = -\frac{2}{13}\text{cosec}^2 z$$

$$f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C \quad \text{[by Milne−Thomson rule]}$$

Where, C is a complex constant $f(z) = \int \left(\frac{-2}{13}\right) \text{cosec}^2 z \, dz - i \int \left(\frac{3}{13}\right) \text{cosec}^2 z dz + C$

$$= \left(\frac{2}{13}\right) \cot z + \left(\frac{3}{13}\right) \cot z + C$$

$$= \frac{2+3i}{i3} \cot z + C$$

Example: 3.35 Find the analytic function $f(z) = u + iv$ given that $2u + 3v = e^x(\cos y - \sin y)$

Solution:

Given $2u + 3v = e^x(\cos y - \sin y)$

Differentiate p.w.r. to x, we get

$$2u_x + 3v_x = e^x(\cos y - \sin y)$$

$$2u_x - 3u_y = e^x(\cos y - \sin y) \quad \text{[by C−R condition]}$$

$$2u_x(z,0) - 3u_y(z,0) = e^z \qquad \ldots(1)$$

Differentiate p.w.r. to y, we get

$$2u_y + 3v_y = e^x[-\sin y - \cos y]$$

$$2u_y + 3u_x = -e^x\,[\sin y + \cos y] \quad \text{[by C−R condition]}$$

$$2u_y(z,0) + 3u_x(z,0) = -e^z \qquad \ldots(2)$$

$$(1) \times (3) \quad \Rightarrow 6u_x(z,0) - 9u_y(z,0) = 3e^z \qquad \ldots(3)$$

$$(2) \times 2 \quad \Rightarrow 6u_x(z,0) + 4u_y(z,0) = -2e^z \qquad \ldots(4)$$

$$(3) - (4) \quad \Rightarrow -13u_y(z,0) = 5e^z$$

$$\Rightarrow u_y(z,0) = -\frac{5}{13} e^z$$

$$(1) \quad \Rightarrow 2u_x(z,0) + \frac{15}{13} e^z = e^z$$

$$2u_x(z,0) = e^z - \frac{15}{13} e^z = -\frac{2}{13} e^z$$

$$\Rightarrow u_x(z,0) = -\frac{1}{13} e^z$$

$$f(z) = \int u_x\,(z,0) dz - i \int u_y(z,0) dz + C$$

$$\therefore f(z) = \int \frac{-1}{13} e^z \, dz - i \int \left(\frac{-5}{13}\right) dz + C$$

$$= \frac{-1}{13} e^z + \frac{5}{13} e^z i + C = \frac{-1+5i}{13} e^z + C$$