Z TRANSFORM AND ITS PROPERTIES

The Z transform of x(n) will convert the time domain signal x(n) to z-domain signal X(z),where the signal becomes a function of complex variables z.

$$X(n)=\sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$x[n]$	$X(z)$	R
1	$\frac{z}{z-1}$	1
$u_1[n]$	$\frac{z}{z-1}$	1
$\delta[n]$	1	0 ($z = 0$ included)
nT	$\frac{Tz}{(z-1)^2}$	1
$(nT)^2$	$\frac{T^2 z(z+1)}{(z-1)^3}$	1
$(nT)^3$	$\frac{T^3 z(z^2+4z+1)}{(z-1)^4}$	1
a^n	$\frac{z}{z-a}$	$\|a\|$
$(n+1)a^n$	$\frac{z^2}{(z-a)^2}$	$\|a\|$
$\frac{(n+1)(n+2)}{2!}a^n$	$\frac{z^3}{(z-a)^3}$	$\|a\|$
$\frac{(n+1)(n+2)(n+3)}{3!}a^n$	$\frac{z^4}{(z-a)^4}$	$\|a\|$
$\frac{(n+1)(n+2)(n+3)(n+4)}{4!}a^n$	$\frac{z^5}{(z-a)^5}$	$\|a\|$
na^n	$\frac{az}{(z-a)^2}$	$\|a\|$
$n^2 a^n$	$\frac{az(z+a)}{(z-a)^3}$	$\|a\|$
$n^3 a^n$	$\frac{az(z^2+4az+a^2)}{(z-a)^4}$	$\|a\|$
$\frac{a^n}{n!}$	$e^{a/z}$	0
e^{-anT}	$\frac{z}{z-e^{-aT}}$	$\|e^{-aT}\|$
$a^n \sin n\omega T$	$\frac{az \sin \omega T}{z^2 - 2az \cos \omega T + a^2}$	$\|a\|$
$a^n \cos n\omega T$	$\frac{z^2 - za \cos \omega T}{z^2 - 2az \cos \omega T + a^2}$	$\|a\|$
$e^{-anT} \sin \omega_0 nT$	$\frac{ze^{-aT} \sin \omega_0 T}{z^2 - 2ze^{-aT} \cos \omega_0 T + e^{-2aT}}$	$\|e^{-aT}\|$

Fig.4.3.1.Table of Z transform pair

[Source: 'Digital Signal Processing Principles, Algorithms and Applications' by J.G. Proakis and D.G. Manolakis page-174]

4.3.1. PROPERTIES OF Z TRANSFORM

1. LINEARITY PROPERTY:

The linearity property of Z transform states that the Z transform of linear weighted combination of discrete time signals is equal to similar weighted combination of Z transform of individual discrete time signals.

Let $z\{x_1(n)\} = X_1(z)$ and

$Z\{x_2(n)\} = X_2(z)$ then by linearity property

$$Z\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 X_1(z) + a_2 X_2(z)$$

Where a_1 and a_2 are constants.

2. SHIFTING PROPERTY:

Case i: Two sided Z transform

The shifting property of Z transform states that Z transform of a shifted signal shifted by m-units of time is obtained by multiplying z^m to Z transform of unshifted signal.

Let $\{x(n)\} = X(z)$

Now, by shifting property,

$$Z\{x(n-m)\} = z^{-m} X(z)$$

$$Z\{x(n+m)\} = z^{m} X(z)$$

Case-ii: One-sided Z transforms

Let x (n) be a discrete time signal defined in the range $0<n<\infty$

Let $\{x(n)\} = X(z)$

Now, by shifting property,

$$Z\{x(n-m)\} = z^{-m} X(z) + \sum_{i=1}^{m} x(-i) z^{-(m-i)}$$

$$Z\{x(n+m)\} = z^{m} X(z) - \sum_{i=0}^{m-1} x(i) z^{(m-i)}$$

3. MULTIPLICATION BY n (OR DIFFERENTIATION IN Z DOMAIN)

Let $\{x(n)\} = X(z)$

Then $Z\{nx(n)\} = -z \frac{d}{dz} X(z)$

In general,

$$Z\{n^m x(n)\} = \left(-z \frac{d}{dz}\right)^m X(z)$$

4. MULTIPLICATION BY AN EXPONENTIAL SEQUENCE, an (OR SCALING IN Z-DOMAIN)

Let{x (n)} =X (z)

Then Z {an x (n)} =X (a-1z)

5. TIME REVERSAL

Let{x (n)} =X (z)

Then Z{x (-n)} = X (z-1)

6. CONJUGATION

If Let, Z{x (n)}=X (z)

Then Z{x*(n)} =X*(z*)

7. CONVOLUTION THEOREM

If Let, Z{x1(n)}=X1(z)

And Z{x2(n)} =X2(z)

Then Z{x1(n)*x2(n)} = X1(z) X2(z)

Where x1(n)*x2(n) = ∑∞m=−∞x1(m)x2(n-m)

8. CORRELATION PROPERTY

If Z{x (n)} =X (z) and Z{y (n)} =Y (z)

Then Z {r_{xy}(m)} =X (z) Y (z-1)

Where rxy (m)=∑∞n=−∞$x(n)y(n-m)$

9. INITIAL VALUE THEOREM

Let x (n) be an one-sided signal defined in the range $0 \le n \le \infty$.

Now, if Z{x (n)} =X (z),

Then the initial value of x (n) [x (0)] is given by,

$$x\ (0)=\lim_{z\to\infty} X(z)$$

10. FINAL VALUE THEOREM

Let x (n) be a one sided signal defined in the range

$0 \le n \le \infty$

Now if Z{x (n)} =X (z),

Then the final value of x (n) is given by

$$x\ (\infty)=\lim_{z\to 1}(1-z-1)X(z)$$

11. COMPLEX CONVOLUTION THEOREM

Let, Z{x1(n)} =X1(z)

And Z{x2(n)} =X2(z)

Now the complex convolution theorem states that,

$$Z\{x1(n)\, x2(n)\} = \frac{1}{2\pi j} \oint X_1(v) X_2\left(\frac{z}{v}\right) v^{-1}\, dv$$

Where, v is a dummy variable used for contour integration.

12. PARSEVAL'S RELATION

If Z{x1(n)} =X1(z) and Z{x2(n)} =X2(z)

Then the Parseval's relation states that

$$\sum_{n=-\infty}^{\infty} x1(n)\, x2^*(n) = \frac{1}{2\pi j} \oint X1(z) X_2^*\left(\frac{1}{z^*}\right) z^{-1}\, dz$$

INVERSE Z TRANSFORM

The inverse z transform is the process of recovering the discrete time signal x (n) from its z transform X (z).

The inverse z transform can be determined by the following three methods.

1. Direct evaluation by contour integration (or residue method)
2. Partial fraction expansion method.
3. Power series expansion method

4.3.2.1 DIRECT EVALUATION BY CONTOUR INTEGRATION (OR RESIDUE METHOD)

Now by the definition of inverse z transform,

$$x(n)=\frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

Using partial fraction expansion technique the function X (z) z^{n-1} can be expressed as

$$X(z)\,z^{n-1} = \frac{A_1}{z-p1}+\frac{A_2}{z-p2}+\frac{A_3}{z-p3}+\ldots+\frac{A_N}{z-pN} \qquad (1)$$

Where p1,p2 are the poles of X(z) and A1,A2 are the residues.

The residue A1 is obtained by multiplying the equation by (z-p1) and letting z=p1

Similarly the residues are evaluated.

$$A1=(z-p_1)\,X(z)\,z^{n-1}\,|_{z=p1}$$

$$A2=(z-p_2)\,X(z)\,z^{n-1}\,|_{z=p2}$$

$$A3=(z-p_3)\,X(z)\,z^{n-1}\,|_{z=p3}$$

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$$A_N=(z-p_N)\,X(z)\,z^{n-1}\,|_{z=pN}$$

The equation (1) can be written as

$$x(n)=\frac{1}{2\pi j}\oint\left[\frac{A_1}{z-p1}+\frac{A2}{z-p2}+\frac{A3}{z-p3}+\ldots+\frac{A_N}{z-pN}\right]dz$$

Using Cauchy's integral theorem

$$x(n)=\sum_{i=1}^{N}[(z-p_i)X(z)z^{n-1}]\,|_{z=pi}$$

4.3.2.2 PARTIAL FRACTION EXPANSION METHOD .

Let X (z) is Z transform of x (n).

$$X(z) = \frac{N(z)}{D(z)}$$

Where N (z) =Numerator polynomial of X (z)

D (z) = denominator polynomial of X (z)

$$\frac{X(z)}{z} = \frac{N(z)}{D(z)z}$$

$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)}$$

On factorizing the denominator polynomial of equation

$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)} = \frac{Q(z)}{(z-p1)(z-p2)(z-p3)..........(z-pN)}$$

Where p1, p2 are the poles of the denominator polynomial

Evaluation of residues:

The coefficients of the denominator polynomial D (z) are assumed real and so the roots of the denominator polynomial are real and or complex conjugate pairs. Hence on factorizing polynomial are poles of X (z).

Case 1: When the roots are real and distinct

In this case $\frac{X(z)}{z}$ *can be expressed as*

$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)} = \frac{Q(z)}{(z-p1)(z-p2)(z-p3)..........(z-pN)}$$

$$= \frac{A1}{z-p1} + \frac{A2}{z-p2} + \ldots\ldots\ldots\ldots + \frac{AN}{z-pN}$$

Where A1, A2………………..AN are the residues

Case 2: When the roots have multiplicity

In this case $\frac{X(z)}{z}$ *can be expressed as*

$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)} = \frac{Q(z)}{(z-p1)(z-p2)(z-p3)..........(z-pN)}$$

$$= \frac{A1}{z-p1} + \frac{A2}{z-p2} + \ldots\ldots\ldots\ldots + \frac{AN}{z-pN}$$

The residue Axr of repeated root is obtained as shown below

$$Axr = \frac{1}{r!} \frac{dr}{dz^r} \left[(z-px)^q \frac{X(z)}{z}\right] \Big|_{z=px} \text{ where } r=0,1,2,\ldots\ldots\ldots(q-1)$$

Case 3: When the roots are complex conjugate

In this case $\frac{X(z)}{z}$ *can be expressed as*

$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)} = \frac{Q(z)}{(z-p_1)(z-p_2)(z-p_3)\ldots\ldots(z^2+az+b)\ldots\ldots(z-p_N)}$$

$$= \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \ldots\ldots + \frac{A_x}{z-(x+iy)} + \frac{A_x^*}{z-(x-iy)} + \ldots\ldots\ldots\ldots + \frac{A_N}{z-p_N}$$

The residues of real and non repeated roots are evaluated as explained as case-1.

4.3.2.3 POWER SERIES EXPANSION METHOD

Let X (z) be z transform of x (n) and X(z) be a rational function of z as shown below

Case-i:

$$X(z) = \frac{N(z)}{D(z)} = c_0 + c_1 z^{-1} + c_2 z^{-2} + c_3 z^{-3} \ldots\ldots \text{---------------------}(1)$$

Case-ii:

$$X(z) = \frac{N(z)}{D(z)} = d_0 + d_1 z^{1} + d_2 z^{2} + d_3 z^{3} + \ldots\ldots \text{---------------------}(2)$$

Case-iii:

$$X(z) = \frac{N(z)}{D(z)} = \ldots\ldots\ldots + e_{-2} z^{2} + e_{-1} z^{1} + e_0 + e_1 z^{-1} + e_2 z^{-2} + \ldots\ldots \text{—}(3)$$

By the definition of Z transform, we get

$$X(n) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

On expanding the summation we get

$$X(z) = \ldots\ldots\ldots\ldots\ldots\ldots x(-3)z^{3} + x(-2)z^{2} + x(-1)z^{1} + x(0)z^{0} + x(1)z^{-1} + \ldots\ldots \text{----}(4)$$

On comparing the coefficients of z of the equations 1 with 4 the samples are determined.