2.2 Continuity

Definition:

A function f is continuous at a number 'a' if $\lim_{x \to a} f(x) = f(a)$

Note: (i)

If f is continuous at a, then

1. $f(a)$should exist.
2. $\lim_{x \to a} f(x)$ exist both on the left and right
3. $\lim_{x \to a} f(x) = f(a)$

The definition says that f is continuous of a if $f(x)$ approaches $f(a)$ as$x$ approaches a.

Note: (ii)

The function $f(x)$ is said to be discontinuous at $x = a$ if one or more of the above three conditions are not satisfied**.**

Example:

Explain why the function is discontinuous at the given number 'a'?

a) $f(x) = \frac{1}{x+2}, a = -2$

b) $f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & if\ x \neq 2 \\ 1 & if\ x = 2 \end{cases}$

c) $f(x) = \begin{cases} \cos x & if \quad x < 0 \\ 0 & if \quad x = 0 \\ 1 - x^2 & if \quad x > 0 \end{cases}$

Solution:

a) Given $f(x) = \frac{1}{x+2}, a = -2$

$$f(-2) = \frac{1}{-2+2} = \frac{1}{0} = \infty, \text{ undefined.}$$

$\therefore f(x)$ is discontinuous at a.

b)$\lim_{x \to a} f(x) = f(a)$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2-x-2}{x-2}$$

$$= \lim_{x \to 2} \frac{(x-2)(x+1)}{x-2}$$

$= \lim_{x\to 2}(x+1) = 3 \neq 1 = f(2)$ given

Hence the function is discontinuous at $x = 2$.

c) The given function is defined for all real values of x.

Check: $\lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \cos x = \cos 0 = 1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(1 - x^2) = 1 - 0 = 1$

But it is given $f(0) = 0$

$\therefore \lim_{x\to 0} f(x) \neq f(0)$

$\therefore f(x)$ is discontinuous at $x = 0$

Example

How would you remove the discontinuity of $f(x) = \frac{x^3-8}{x^2-4}$

Solution:

$\lim_{x\to a} f(x) = f(a)$

Given $f(x) = \frac{x^3-8}{x^2-4}$

$f(x)$ is defined in all the real vlues except at $x = 2$.

$\therefore f(2)$ is not defined.

$$\text{But } \lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{x^3-8}{x^2-4} = \lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{(x+2)(x-2)}$$

$$= \lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{(x+2)(x-2)}$$

$$= \lim_{x\to 2}\frac{(x^2+2x+4)}{(x+2)}$$

$$= \frac{4+4+4}{2+2} = \frac{12}{4} = 3$$

Then the discontinuity is removed.

$$\therefore \text{The function is defined as} \begin{cases} \frac{x^3-8}{x^2-4} & if\ x \neq 2 \\ 3 & if\ x = 2 \end{cases}$$

Theorem:

If f and g are continuous at 'a' and c is a constant then the following are continuous.

(i) cf (ii) $f \pm g$ (iii) fg (iv) $\frac{f}{g}$ if $g(a) \neq 0$

Theorem:

(i) Any polynomial is continuous everywhere.

(ii) Any rational function is continuous wherever it is defined.

Note:

The following types of functions are continuous at every number in their domains: Polynomials, rational fractions, root functions, trigonometrical functions, inverse trigonometric functions, exponential functions, logarithmic functions.

Theorem:

$\lim_{x\to a} f\big(g(x)\big) = f(\lim_{x\to a} g(x))$

Theorem:

If g is continuous at a and f is continuous at $g(a)$, then the composite function fog

given by $(fog)x = f(g(x))$ is continuous at a.

Theorem: (The Intermediate value theorem):

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number

between $f(a)$ and $f(b)$,where $f(a) \neq f(b)$

Then there exists a number c in (a,b) Such that $f(c) = N$.

Example.

Discuss the continuity of the function $\frac{x^2-x-2}{x-2}$

Solution:

A function f is continuous at 'a' if $\lim_{x\to a} f(x) = f(a)$

The given function $\frac{x^2-x-2}{x-2}$ is defined for all real value of x except at $x = 2$.

So $f(2)$ is not defined.

Hence the function is discontinuous at $x = 2$.

Example

Evaluate $\lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right)(or)\lim_{x\to 1} arc\ \sin\frac{1-\sqrt{x}}{1-x}$

Solution:

$$\lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right) = \lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-(\sqrt{x})^2}\right)$$

$$= \lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{(1+\sqrt{x})(1-\sqrt{x})}\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Example

Show that the junction f(x) = 1 - $\sqrt{1-x^2}$ is continuous on the interval [-1, 1]

Solution:

Given $f(x)$ = 1 - $\sqrt{1-x^2}$ in [-1, 1]

Let $a \epsilon$ [-1, 1] , i.e., $-1 < a < 1$

To Prove $lim_{x\to a} f(x) = f(a)$

L.H.S = $lim_{x\to a} f(x)$

$= lim_{x\to a}[1-\sqrt{1-x^2}]$

$= 1 - \sqrt{1-a^2}$

$= f(a)$

∴ The given function is continuous.

Example

For what value of the constant b is the function f continuous on (-∞, ∞)

$$f(x) = \begin{cases} bx^2 + 2x & if \quad x < 2 \\ x^3 - bx & if \quad x \geq 2 \end{cases}$$

Solution:

Given the function is continuous.

∴ $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

$\lim_{x\to 2} (bx^2 + 2x) = \lim_{x\to 2}(x^3 - bx)$

$\Rightarrow 4b + 4 = 8 - 2b$

$\Rightarrow 4b + 2b = 8 - 4$

$\Rightarrow 6b = 4$

$\Rightarrow b = \frac{4}{6} = \frac{2}{3}$

Example

Suppose f and g are continuous functions such that $g(2) = 6$ and $\lim_{x\to 2}[3f(x) + f(x)g(x)] = 36$, Find $f(2)$

Solution:

Given, $\lim_{x\to 2}[3f(x) + f(x)g(x)] = 36$, $g(2) = 6$

$3\lim_{x\to 2} f(x) + \lim_{x\to 2} f(x)g(x) = 36$

$\Rightarrow 3f(2) + f(2)g(2) = 36$

$\Rightarrow f(2)[3 + g(2)] = 36$

$\Rightarrow f(2)[3 + 6] = 36$

$\Rightarrow f(2) = \frac{36}{9}$

$\Rightarrow f(2) = 4$

Exercise

1. Find $\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x}$ **Ans:** $\frac{-1}{11}$
2. Use continuity to evaluate $\lim_{x\to\pi} \frac{\sin x}{2+\cos x}$ **Ans:** 0
3. Using continuity to evaluate $\lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right)$ **Ans:** $\frac{\pi}{6}$
4. Use continuity to evaluate $\lim_{x\to 4} \frac{5+\sqrt{x}}{\sqrt{5+x}}$ **Ans:** $\frac{7}{3}$
5. Using continuity to evaluate $\lim_{x\to 2} \tan^{-1}\left(\frac{x^2-4}{3x^2-6x}\right)$ **Ans:** $\tan^{-1}\left(\frac{2}{3}\right)$
6. How would you remove the discontinuity of $f(x) = \frac{x^4-1}{x-1}$ **Ans:** $f(x) =$

$$\begin{cases} \frac{x^4-1}{x-1} & if \quad x \neq 1 \\ 4 & if \quad x = 1 \end{cases}$$

7. Where is the function $f(x) = \frac{\log x + \tan^{-1}x}{x^2-1}$ continuous?

Ans: It is continuous on (0, 1) and (1,∞)

8. Where is the function $f(x) = \sin x^3$ continuous? **Ans:** Continuous on R
9. Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2

Derivatives

Definition:

A real valued function f defined on an open interval I is said to be differentiable at $a \in I$ if $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists and is finite, and the value of the limit is denoted by $f'(a)$ and is called the differential coefficient or the derivative of $f(x)$ at the point $x = a$

If f is differentiable at each point of I, we say that f is differentiable on I.

i.e., $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

Note:

The tangent line to $y = f(x)$ at (x_1, y_1) is the line through (x_1, y_1) whose slope is equal to $f'(a)$, the derivative of f at a.

(i) Equation of tangent line at (x_1, y_1) is $y - y_1 = m(x - x_1)$

(ii) Equation of the normal line at (x_1, y_1) is $y_1 = \left(\frac{-1}{m}\right)(x - x_1)$

Example:

Find an equation of the tangent line to the curve at the given point

a) $\boldsymbol{y = \frac{3}{x}}$ **at (3, 1)** **b)** $\boldsymbol{y = x^2 - 8x + 9}$ **at (3, -6)**

c) $\boldsymbol{y = \sqrt{x}}$ **at (1, 1)**

Solution:

a) $y = f(x) = \frac{3}{x}$ at (3, 1)

$f'(x) = \lim_{x\to 3} \frac{f(x)-f(3)}{x-3}$

$= \lim_{x\to 3} \frac{\frac{3}{x}-1}{x-3}$

$= \lim_{x\to 3} \frac{3-x}{x(x-3)}$

$= \lim_{x\to 3} \left(\frac{-1}{x}\right)$

$= \frac{-1}{3}$

$\therefore$ m $= \frac{-1}{3}$

Equation of the tangent line at (3, 1) is

$y - y_1 = m\,(x - x_1)$

$\Rightarrow y - 1 = \frac{-1}{3}\,(x - 3)$

$\Rightarrow 3y - 3 = -x + 3$

$\Rightarrow x + 3y - 6 = 0$

$y = \frac{6-x}{3}$

y = $\frac{-1}{3}$x + 2

b) $y = x^2 - 8x + 9$at (3, -6)

$f'\,(x) = \lim_{x\to 3} \frac{f\,(x)-f(3)}{x-3}$

$= \lim_{x\to 3} \frac{x^2-8x+9-(-6)}{x-3}$

$= \lim_{x\to 3} \frac{x^2-8x+15}{x-3}$

$= \lim_{x\to 3} \frac{(x-5)(x-3)}{x-3}$

$= \lim_{x\to 3} (x-5)$

$m = -2$

Equation of the tangent line at (3, -6) is

$y - y_1 = m\,(x - x_1)$

$y + 6 = -2\,(x - 3)$

$y = -2x + 6 - 6$

$y = -2x$

c) $y = \sqrt{x}$ at (1, 1) $y = f(x) = \sqrt{x}$

$$f'(x) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$$

$$= \lim_{x\to 1} \frac{\sqrt{x}-1}{(\sqrt{x}+1)(\sqrt{x}-1)}$$

$$= \lim_{x\to 1} \frac{1}{\sqrt{x}+1}$$

m = $\frac{1}{2}$

Equation of the tangent line at (1, 1) is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{1}{2}(x - 1)$$

$$2y - 2 = x - 1$$

$$2y = \frac{x+1}{2}$$

Example:

If $f(x) = x^3 - x$, find $f'(x)$ and also find $f''(x)$

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[(x+h)^3-(x+h)]-(x^3-x)}{h}$$

$$= \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x-h-x^3+x}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3-h}{h}$$

$$= \lim_{h\to 0} 3x^2 + 3xh + h^2 - 1$$

$$f'(x) = 3x^2 - 1$$

$$f''(x) = \lim_{h\to 0} \frac{f'(\mathrm{x+h})-f'(\mathrm{x})}{h}$$

$$= \lim_{h\to 0} \frac{[3(x+h)^2-1]-[3x^2-1]}{h}$$

$$= \lim_{h\to 0} \frac{3x^2+6xh+3h^2-1-3x^2+1}{h}$$

$$= \lim_{h\to 0} \frac{6xh+3h^2}{h}$$

$$= \lim_{h\to 0} 6x + 3h$$

$f''(x) = 6x$

Example

If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'

Solution:

$$f(x) = \sqrt{x}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x}+h-\sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x}+h+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$f'(x)\ exists\ if\ x > 0$

Hence the domain of f' is $(0, \infty)$

This is smaller than the domain of f, which is $[0, \infty)$

Exercise

1. Find the equation of the tangent line to the curve at the given point

(i) $f(x) = 4x - 3x^2$ at (2, -4) **Ans:** $y = -8x + 12$

(ii) $f(x) = x^4$ at (1, 0) **Ans:** $y = 4x - 4$

(iii) $f(x) = 3x^2 - x^3$ at (1, 2) **Ans:** $3x - 1$

2. Find the derivative of the function $f(x) = x^3 - 7x$, and find the equation of $f'(a)$

Ans: $f'(a) = 3a^2 - 7$

3. Find the derivative if the following functions:

(i) $f\ (x) = x^3 - 3x\ +\ 5$ **Ans:** $f'\ (x) = 3x^2 - 3$

(ii) $f\ (x) = x^n$ **Ans:** $f'\ (x) = nx^{n-1}$

(iii) $f\ (x) = e^x$ **Ans:** $f'\ (x) = e^x$