Method of variation of parameters:

This method is very useful in finding the general solution of the second order equation

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = x$$

Solution is $y = Au + Bv$

Where A, B are constants

$$A = \int \frac{-Rv}{uv' - vu'}\, dx + C_1$$

$$B = \int \frac{Ru}{uv' - vu'}\, dx + C_2$$

Problems Based on Method of variation of parameters

Example:1

Solve $\frac{d^2y}{dx^2} + a^2y = \sec ax$

Solution:

$(D^2 + a^2)y = \sec ax$

Auxiliary Equation is $m^2 + a^2 = 0$

$$m = \pm ia$$

C.F = $(A\cos ax + B\sin ax)$

Let the solution be $y = Au + Bv$

Here

u = cos ax	v= sin ax

$$u' = -\operatorname{asin} ax \qquad v' = a\cos ax$$

$$uv' - vu' = a\, cos^2\, ax + a\, sin^2\, ax \;=\; a$$

$$A = \int \frac{-Rv}{uv'-vu'}\, dx + C_1$$

$$= \int \frac{-\sec ax \sin ax}{a}\, dx + C_1$$

$$= \frac{-1}{a}\int \frac{1}{\cos ax}\sin ax\, dx + C_1$$

$$= \frac{-1}{a}\int \tan ax\; dx + C_1$$

$$= +\frac{\log\cos ax}{a^2} + C_1$$

$$B = \int \frac{Ru}{uv'-vu'}\, dx + C_2$$

$$= \int \frac{\sec ax \cos ax}{a}\, dx + C_2$$

$$= \frac{1}{a}\int \frac{1}{\cos ax}\cos ax\, dx + C_2$$

$$= \frac{1}{a}x + C_2$$

$$\therefore y = Au + Bv$$

$$y = \cos ax\left[\frac{\log(\cos ax)}{a^2} + C_1\right] + \sin ax\left[\frac{x}{a} + C_2\right]$$

$$y = C_1\cos ax + \cos ax\,\frac{\log(\cos ax)}{a^2} + C_2\sin ax + \frac{x}{a}\sin ax$$

Example:2

Solve $(D^2+4)y = \tan 2x$

Solution:

Auxiliary Equation is $m^2 + 4 = 0$

$m^2 = -4$

$m = \pm 2i$

C.F = $A\, cos\, ax + B \sin ax$

Let the solution be $y = Au + Bv$

$u = cos\, 2x \qquad v = sin\, 2x$

$u' = -2\sin 2x \qquad v' = 2\cos 2x$

$uv' - vu' = 2\, cos^2\, 2x + 2\, sin^2\, 2x$

$= 2$

$A = \int \frac{-Rv}{uv'-vu'}\, dx + C_1$ R= tan 2x

$= \int \frac{-\tan 2x \sin 2x}{2}\, dx + C_1$

$= \frac{-1}{2} \int \frac{\sin 2x}{\cos 2x} \sin 2x\, dx + C_1$

$= \frac{-1}{2} \int \frac{\sin^2 2x}{\cos 2x} dx + C_1$

$= \frac{-1}{2} \int \frac{(1-cos^2 2x)}{\cos 2x} dx + C_1$

$= \frac{-1}{2} \int \left(\frac{1}{\cos 2x} - \frac{cos^2\, 2x}{\cos 2x}\right) dx + C_1$

$= \frac{-1}{2} \int (\sec 2x - \cos 2x)\, dx + C_1$

$= \frac{-1}{2} \left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right] + C_1$

$$B = \int \frac{-Ru}{uv'-vu'}\, dx + C_2$$

$$= \int \frac{\tan 2x \cos 2x}{2}\, dx + C_2$$

$$= \int \frac{1}{2} \sin 2x\, dx + C_2$$

$$= -\frac{1}{2}\frac{\cos 2x}{2} + C_2$$

$$= -\frac{\cos 2x}{4} + C_2$$

$$\therefore y = Au + Bv$$

$$y = \left\{\left[\frac{-1}{4}(\log \sec 2x + \tan 2x) - \frac{\sin 2x}{4}\right] + C_1\right\}\cos 2x + \left(-\frac{\cos 2x}{4} + C_2\right)\sin 2x$$

Example: 3

Solve $y'' - 2y' = e^x \sin\, x$

Solution:

$$(D^2 + 2D)y = e^x \sin x$$

Auxiliary Equation is $m^2 - 2m = 0$

$$m(m-2) = 0$$

$$m = 0, m = 2$$

C.F = $(A\, e^{0x} + Be^{2x})$

Let the solution be y = $Au + Bv$

Here $u = 1$	$v = e^{2x}$
$u' = 0$	$v' = 2e^{2x}$

$$uv' - vu' = 2e^{2x}$$

$$A = \int \frac{-Rv}{uv'-vu'}\, dx + C_1$$

$$= \int \frac{-e^{x}\sin x\,(e^{2x})}{2e^{2x}}\, dx + C_1$$

$$= \frac{-1}{2}\int e^{x}\sin x\, dx + C_1$$

$$= \frac{-1}{2}\left[\frac{e^{x}}{1+1}(sin\ x - \cos x)\right] + C_1$$

$$= \frac{-e^{x}}{4}(\sin x - \cos x) + C_1$$

$$B = \int \frac{Ru}{uv'-vu'}\, dx + C_2$$

$$= \int \frac{e^{x}\sin x}{2e^{2x}}\, dx + C_2 \ \left\{ \int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - bcos\ bx]\right\}$$

$$= \frac{1}{2}\int e^{-x}\sin x\, dx + C_2$$

$$= \frac{1}{2}\,\frac{e^{-x}}{2}(-\sin x - \cos x) + C_2$$

$$= \frac{-e^{-x}}{4}(\sin x + \cos x) + C_2$$

$$\therefore y = Au + Bv$$

$$y = \left[\frac{-e^{x}}{4}(\sin x - \cos x) + C_1\right] + \left[\frac{-e^{-x}}{4}(\sin x + \cos x) + C_2\right]e^{2x}$$

Example:4

Solve $y'' + 2y' + y = e^{-x}\log\ x$

Solution:

$$(D^2 + 2D + 1)y = e^{-x}\log x$$

Auxiliary Equation is $m^2 + 2m + 1 = 0$

$$(m+1)^2 = 0$$

$$m + 1 = 0 \quad \text{(twice)}$$

$$m = -1 \quad \text{(twice)}$$

$$m = -1, -1$$

C.F = $(A + Bx)e^{-x}$

$= (Ae^{-x} + Bxe^{-x})$

Let the solution be $y = Au + Bv$

Here $u = e^{-x}$ | $v = xe^{-x}$

$u' = -e^{-x}$ | $v' = -xe^{2x} + e^{-x}$

$$uv' - vu' = -xe^{-2x} + e^{-2x} + xe^{-2x}$$

$$= e^{-2x}$$

$$A = \int \frac{-Rv}{uv' - vu'} \, dx + C_1$$

$$= \int \frac{-e^{-x}\log x}{e^{-2x}} xe^{-x} \, dx + C_1$$

$$= -\int x \log x \, dx + C_1$$

Here u = $\log x$ | $\int dv = \int x \, dx$

$u' = \frac{1}{x} \, dx$ | $v = \frac{x^2}{2}$

$$= -\left[\log x \left(\frac{x^2}{2}\right) - \int \frac{x^2}{2}\frac{1}{x} \, dx\right]$$

$$= -\left[\frac{x^2}{2}\log x - \frac{x^2}{4} + C_1\right]$$

$$B = \int \frac{Ru}{uv' - vu'}\, dx + C_2$$

$$= \int \frac{e^{-x}\log x}{e^{-2x}} e^{-x}\, dx + C_2$$

$$= \int \log x\, dx + C_2$$

$$= x\log x - x + C_2$$

$$\therefore y = Au + Bv$$

$$y = \left[\frac{-x^2}{2}\log x - \frac{x^2}{2} + C_1\right]e^{-x} + (x\log x - x + C_2)xe^{-x}$$

Example: 5

Solve by method of variation of parameter $\frac{d^2y}{dx^2} + y = x\sin x$

Solution:

$$(D^2 + 1)y = x\sin x$$

Auxiliary Equation is $m^2 + 1 = 0$

$$m = \pm i$$

C.F = $(A\cos x + B\sin x)$

Let the solution be $y = Au + Bv$

Here $u = \cos x$ | $v = \sin x$

$u' = -\sin x$ | $v' = \cos x$

$$uv' - vu' = cos^2x + sin^2x$$

$$= 1 \quad ; \qquad \text{R} = x\, sin\, x$$

$$A = \int \frac{-Rv}{uv'-vu'}\, dx + C_1$$

$$= \int \frac{-x \sin x \sin x}{1}\, dx + C_1$$

$$= -\int \frac{x(1-\cos 2x)}{2}\, dx + C_1$$

$$= \frac{-1}{2}\left[\frac{x^2}{2}\right] + \frac{1}{2}\left[\frac{x sin\, 2x}{2} - (1)\left(\frac{-\cos 2x}{4}\right)\right] + C_1$$

$$= \frac{-x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x + C_1$$

$$B = \int \frac{Ru}{uv'-vu'}\, dx + C_2$$

$$= \int \frac{x \cos x \sin x}{1} dx$$

$$= \int \frac{x \sin 2x}{2} dx$$

$$= \frac{1}{2}\left[x\left(\frac{-\cos 2x}{2}\right) - (1)\left(\frac{-\sin 2x}{4}\right)\right] + C_2$$

$$= \frac{-x}{4}\cos 2x + \frac{1}{8}\sin 2x + C_2$$

$$\therefore\ y = Au + Bv$$

$$y = \left[\frac{-x^2}{2} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x + C_1\right]\cos x + \left[\frac{-x}{4}\cos 2x + \frac{1}{8}\sin 2x + C_2\right]\sin x$$

Example:6

Solve $(D^2 + 2D + 1)\, y = \frac{e^{-x}}{x^2}$ by variation parameter

Solution:

Auxiliary Equation is $m^2 + 2m + 1 = 0$

$$m = -1, -1$$

$$\text{C.F} = (A + Bx)e^{-x}$$

$$= A\, e^{-x} + Bxe^{-x}$$

Let the solution be $y = Au + Bv$

Here $u = e^{-x}$ | $v = xe^{-x}$

$u' = -e^{-x}$ | $v' = -xe^{-x} + e^{-x}$

$$uv' - vu' = -xe^{-2x} + e^{-2x} + xe^{-2x}$$

$$= e^{-2x} \quad ; \quad R = \frac{e^{-x}}{x^2}$$

$$A = \int \frac{-Rv}{uv' - vu'}\, dx + C_1$$

$$= \int \frac{\frac{e^{-x}}{x^2}\ xe^{-4}}{e^{-2x}}\, dx + C_1$$

$$= -\int \frac{xe^{-2x}}{x^2 e^{-2x}}\, dx + C_1$$

$$= -\int \frac{1}{x}\, dx + C_1$$

$$= -\log x + C_1$$

$$B = \int \frac{Ru}{uv' - vu'}\, dx + C_2$$

$$= \int \frac{\frac{e^{-x}}{x^2} \; e^{-x}}{e^{-2x}} \, dx + C_2$$

$$= \int \frac{1}{x^2} dx + C_2$$

$$= \frac{-1}{x} + C_2$$

$$\therefore \quad y = Au + Bv$$

$$y = (-\log x + C_1)e^{-x} + \left(\frac{-1}{x} + C_2\right) x e^{-x}$$

Example:7

Solve the equation $\boldsymbol{\frac{d^2y}{dx^2} + y = cosec\, x}$

Solution:

$$(D^2 + 1)y = cosec\, x$$

Auxiliary Equation is $m^2 + 1 = 0$

$$m^2 = -1$$

$$m = \pm i$$

$$\alpha = 0, \beta = 1$$

C.F = $A\cos x + B\sin x$

Let the solution be $y = Au + Bv$

Here $u = \cos x$ | $v = \sin x$

$u' = -\sin x$ | $v' = \cos x$

$$uv' - vu' = \cos^2 x + \sin^2 x$$

$= 1 \quad ; \qquad R = \text{cosec } x$

$$A = \int \frac{-Rv}{uv'-vu'}\, dx + C_1$$

$$= \int \frac{-\sin x \operatorname{cosec} x}{1}\, dx + C_1$$

$$= \int -\, dx + C_1$$

$$= -x + C_1$$

$$B = \int \frac{Ru}{uv'-vu'}\, dx + C_2$$

$$= \int \frac{\operatorname{cosec} x \cos x}{1}\, dx + C_2$$

$$= \int \frac{\cos x}{\sin x} dx + C_2$$

$$= \int \cot x\, dx + C_2$$

$$= \log \sin x + C_2$$

$$\therefore \quad y = Au + Bv$$

$$y = (-x + C_1)\cos x + (\log \sin x + C_2)\sin x$$

Example:8

Solve $(D^2 + 4)y = \cot 2x$

Solution:

Auxiliary Equation is $m^2 + 4 = 0$

$$m^2 = -4$$

$$m = \pm 2i$$

C.F = $A\cos 2x + B\sin 2x$

Let the solution be $y = Au + Bv$

Here $u = cos\,2x$ | $v = sin\,2x$

$u' = -2\sin 2x$ | $v' = 2\cos 2x$

$$uv' - vu' = 2cos^2 2x + 2sin^2 2x$$

$$= 2 \quad ; \qquad \text{R = cot 2x}$$

$$A = \int \frac{-Rv}{uv'-vu'}\, dx + C_1$$

$$= \int \frac{-\cot 2x \sin 2x}{2}\, dx + C_1$$

$$= -\frac{1}{2}\int \cos 2x\, dx + C_1$$

$$= -\frac{1}{2}\left(\frac{\sin 2x}{2}\right) + C_1$$

$$= \frac{-\sin 2x}{4} + C_1$$

$$B = \int \frac{Ru}{uv'-vu'}\, dx + C_2$$

$$= \int \frac{\cot 2x \cos 2x}{2}\, dx + C_2$$

$$= \frac{1}{2}\int \frac{\cos^2 2x}{\sin 2x}\, dx + C_2$$

$$= \frac{1}{2}\int \left(\frac{1-sin^2 2x}{\sin 2x}\right) dx + C_2$$

$$= \frac{1}{2}\int (\text{cosec 2x} - \text{sin 2x})\, dx + C_2$$

$$= \frac{1}{2}\left[-\frac{1}{2}\log(cosec\,2x + \cot 2x) + \frac{1}{2}\cos 2x\right] + C_2$$

$$= -\frac{1}{4}\log(cosec\,2x + \cot 2x) + \frac{1}{4}\cos 2x + C_2$$

$$\therefore \quad y = Au + Bv$$

$$y = \left[-\frac{\sin 2x}{4} + C_1\right]\cos 2x + \left[-\frac{1}{4}\log(cosec\ 2x + \cot 2x) + \frac{1}{4}\cos 2x + C_2\right]\sin 2x$$