1.2 CO- ORDINATE SYSTEMS

A point or vector can be represented in any curvilinear coordinate system, which may be orthogonal or non orthogonal.

An **orthogonal system** is one in which the coordinates are mutually perpendicular.

To describe a vector accurately and to express a vector in terms of its components, it is necessary to have some reference directions. Such directions are represented in terms of various co-ordinate systems. There are various co-ordinates systems available in mathematics, the co-ordinate systems are

- **Cartesian or Rectangular co-ordinate system**
- **Cylindrical co-ordinate system**
- **Spherical co-ordinate system**

CARTESIAN OR RECTANGULAR CO-ORDINATE SYSTEM

There are three simple methods to describe a vector accurately such as specific lengths, directions, angles, projections or components. The simplest methods of these are Cartesian or Rectangular co-ordinate system.

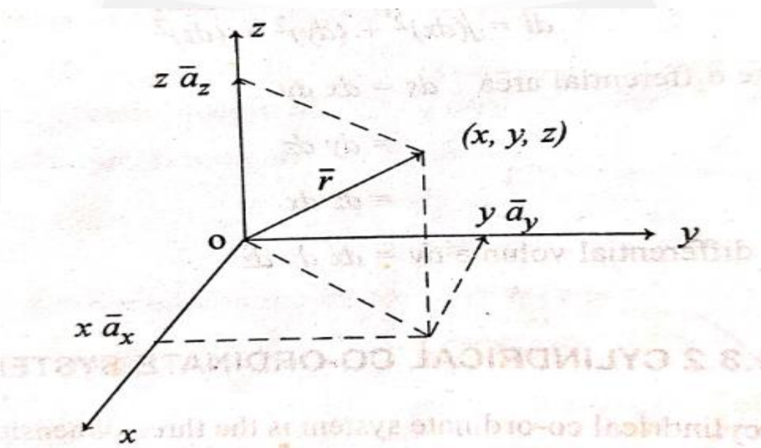

Figure 1.2.1 Cartesian co-ordinate system

[Source: "Elements of Electromagnetics" by Matthew N.O.Sadiku, page-30]

In Cartesian co-ordinate system three co-ordinate axes ***x, y, z*** are mutually right angles to each other. Considered a point **P(*x, y, z*)** in space at a distance **r** from the orgin. The vector **r** can be represented as

$$\vec{r} = x\overrightarrow{a_x} + y\overrightarrow{a_y} + z\overrightarrow{a_z}$$

$\overrightarrow{a_x}, \overrightarrow{a_y}, \overrightarrow{a_z}$ are Unit Vector

x, y, z are the components vectors. Components vectors have a magnitude and direction. Unit vectors have unit magnitude and directed along the co-ordinate axis.

A Unit Vector in a given direction is a vector in that direction divided by its magnitude. It is given by

$$\overrightarrow{a_r} = \frac{\vec{r}}{|r|}$$

$$|r| = \sqrt{x^2 + y^2 + z^2}$$

$$\overrightarrow{a_r} = \frac{x\vec{a}_x + y\vec{a}_y + z\vec{a}_z}{\sqrt{x^2 + y^2 + z^2}}$$

The ranges of the co-ordinate variables ***x, y, z*** are

$$-\infty < x < \infty$$

$$-\infty < y < \infty$$

$$-\infty < z < \infty$$

Considered the points **P(*x, y, z*)** and **Q(*x*+*dx*, *y*+*dy*, *z*+*dz*)** in a rectangular co-ordinate system.

The differential length ***dl*** from **P** to **Q** is the diagonal of the parallel piped is given by

The differential length $\quad dl = \sqrt{(d_x)^2 + (d_y)^2 + (d_z)^2}$

The differential area $\quad d_s = d_x d_y$

$$d_s = d_y d_z$$

$$d_s = d_z d_x$$

The differential Volume $\quad d_v = d_x d_y d_z$

CYLINDRICAL CO-ORDINATE SYSTEM

The circular cylindrical co-ordinate system is very convenient whenever we are dealing with problems having cylindrical symmetry.

A point **P** is cylindrical co-ordinates is represented as (ρ, φ, z) and is shown in figure 1.2.2.

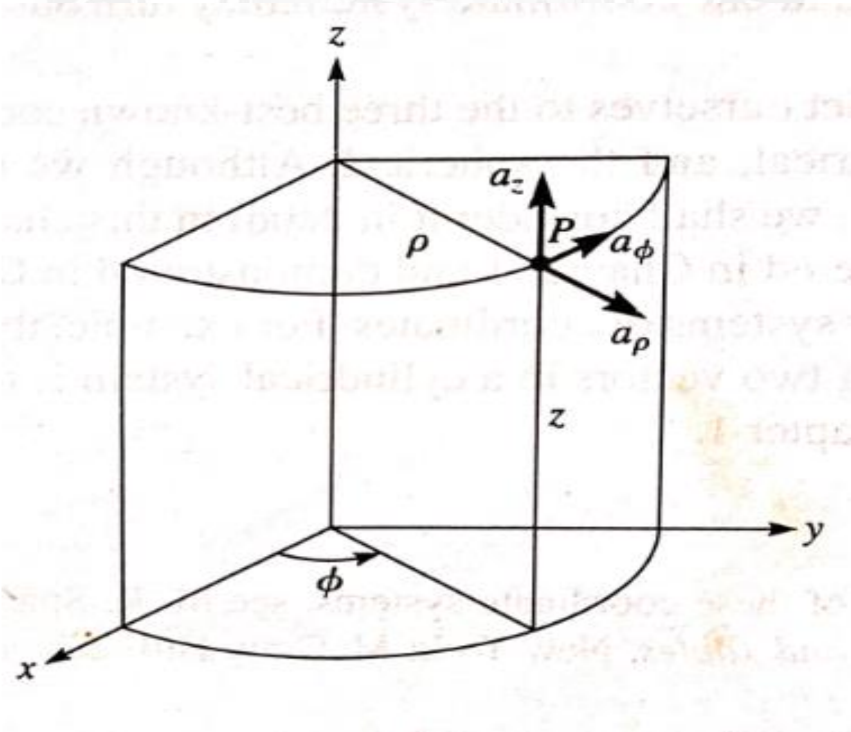

Figure 1.2.2 Point P and unit vector in the cylindrical co-ordinate system

[Source: "Elements of Electromagnetics" by Matthew N.O.Sadiku, page-30]

- $\boldsymbol{\rho}$ is the radius of cylinder passing through **P** or the radial distance from the $\boldsymbol{z}$ - axis
- $\boldsymbol{\varphi}$ called the azimuthal angle, is measured from the $\boldsymbol{x}$ – axis in the xy – plane
- $\boldsymbol{z}$ is the same as in the Cartesian system .

Considered any point as the intersection of three mutually perpendicular surfaces. They is a circular cylinder **(ρ =constant),** a place **(φ =constant),** and another place **(z =constant)**

A differential volume element in cylindrical co-ordinate may obtained by increasing $\boldsymbol{\rho}, \boldsymbol{\varphi}$ and $\boldsymbol{z}$ by the differential increments $\boldsymbol{d\rho}, \boldsymbol{d\varphi}$ and $\boldsymbol{dz}$ .The shape of this small volume is truncated wedge .As the volume element becomes very small.its shape approaches that of a rectangular parallel piped. It has sides of the length $\boldsymbol{d\rho}, \boldsymbol{\rho d\varphi}$ and $\boldsymbol{dz}$

The differential length $\boldsymbol{dl} = \sqrt{(\boldsymbol{d\rho})^2 + (\boldsymbol{\rho d\varphi})^2 + (\boldsymbol{dz})^2}$

The differential area $\boldsymbol{d_s} = \boldsymbol{d\rho} . \boldsymbol{\rho d\varphi} = \boldsymbol{\rho d\rho d\varphi}$

$$\boldsymbol{d_s} = \boldsymbol{\rho d\varphi} . \boldsymbol{dz}$$

$$\boldsymbol{d_s} = \boldsymbol{dz} . \boldsymbol{d\rho} = \boldsymbol{d\rho dz}$$

The differential Volume $\boldsymbol{d_v} = \boldsymbol{d\rho} . \boldsymbol{\rho d\varphi} . \boldsymbol{dz}$

$$\boldsymbol{d_v} = \boldsymbol{\rho d\rho d\varphi dz}$$

The ranges of the co-ordinate variables are

$$\mathbf{0} \leq \boldsymbol{\rho} < \infty$$

$$\mathbf{0} \leq \boldsymbol{\varphi} < 2\boldsymbol{\pi}$$

$$-\infty < z < \infty$$

Now the unit vector $\boldsymbol{a_\rho}, \boldsymbol{a_\varphi}$ and $\boldsymbol{a_z}$ are mutually perpendicular because our co-ordinate system is orthogonal: $\boldsymbol{a_\rho}$ points in the direction of increasing $\boldsymbol{\rho}$, $\boldsymbol{a_\varphi}$ in the direction of increasing $\boldsymbol{\varphi}$ **and** $\boldsymbol{a_z}$ in the positive $\boldsymbol{z}$ - direction

$$\boldsymbol{a_\rho} \cdot \boldsymbol{a_\rho} = \boldsymbol{a_\varphi} \cdot \boldsymbol{a_\varphi} = \boldsymbol{a_z} \cdot \boldsymbol{a_z} = 1$$

$$\boldsymbol{a_\rho} \cdot \boldsymbol{a_\varphi} = \boldsymbol{a_\varphi} \cdot \boldsymbol{a_z} = \boldsymbol{a_z} \cdot \boldsymbol{a_\rho} = 0$$

$$\boldsymbol{a_\rho} \times \boldsymbol{a_\varphi} = \boldsymbol{a_z}$$

$$\boldsymbol{a_\varphi} \times \boldsymbol{a_z} = \boldsymbol{a_\rho}$$

$$\boldsymbol{a_Z} \times \boldsymbol{a_\varphi} = \boldsymbol{a_\varphi}$$

SPHERICAL CO-ORDINATE SYSTEM

The spherical co-ordinate system is most appropriate when one is dealing with problems having a degree of spherical symmetry. A point **P** can be represented as $(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\varphi})$ and it is illustrated in figure 1.2.3.

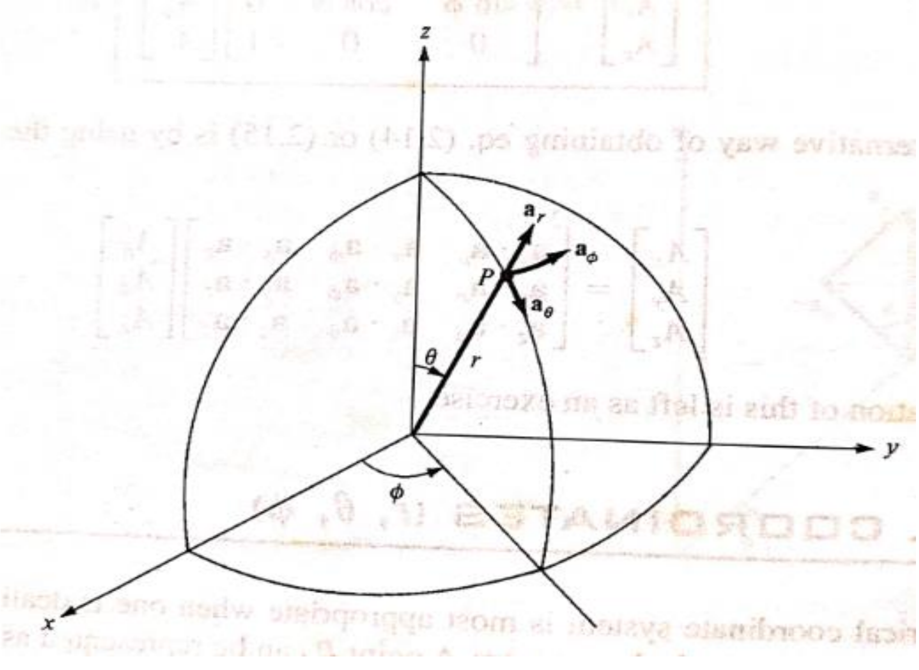

Figure 1.2.3 Point P and unit vector in the spherical co-ordinate

[Source: "Elements of Electromagnetics" by Matthew N.O.Sadiku, page-34]

- **r** is defines as the distance from the origin to **P** or the radius of a sphere entered at the origin and passing through **P .**

- $\boldsymbol{\theta}$ (Called the colatitudes) is the angle between the $\boldsymbol{z}$ – axis and position vector of **P.**

- $\boldsymbol{\varphi}$ is measured from the $\boldsymbol{x}$ **-** axis (the same azimuthal angle in cylindrical co-ordinates)

Considered any point as the intersection of the spherical surfaces **($\boldsymbol{radius\ r}$ =constant),** conical surface(**$\boldsymbol{\theta}$,** angle between ***r and z*****=constant),** and plane surface **($\boldsymbol{\varphi}$ =constant).**The co-ordinates of spherical system are $\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\varphi}$

A differential volume element may be obtained in spherical co-ordinate by increasing $\boldsymbol{r}, \boldsymbol{\theta}$ and $\boldsymbol{\varphi}$ by $\boldsymbol{dr}, \boldsymbol{d\theta}\ \boldsymbol{and}\ \boldsymbol{d\varphi}$ the sides of this volume element are $\boldsymbol{dr}, \boldsymbol{rd\theta}\ \boldsymbol{and}\ \boldsymbol{r}\sin\boldsymbol{\theta d\varphi}.$

The differential length $\quad \boldsymbol{dl} = \sqrt{(\boldsymbol{dr})^2 + (\boldsymbol{rd\theta})^2 + (\boldsymbol{r}\sin\boldsymbol{\theta d\varphi})^2}$

The differential area $\quad \boldsymbol{d_s} = \boldsymbol{dr}.\boldsymbol{rd\theta} = \boldsymbol{r\ dr\ d\theta}$

$$\boldsymbol{d_s} = \boldsymbol{rd\theta}.\boldsymbol{r}\sin\boldsymbol{\theta d\varphi} = \boldsymbol{r}^2\sin\boldsymbol{\theta\ d\theta\ d\varphi}$$

$$\boldsymbol{d_s} = \boldsymbol{r}\sin\boldsymbol{\theta d\varphi}\,.\,\boldsymbol{dr} = \boldsymbol{r}\sin\boldsymbol{\theta d\varphi\ \ dr}$$

The differential Volume $\quad \boldsymbol{d_v} = \boldsymbol{dr}.\boldsymbol{rd\theta}.\boldsymbol{r}\sin\boldsymbol{\theta d\varphi}$

$$\boldsymbol{d_v} = \boldsymbol{r}^2\sin\boldsymbol{\theta\ d\theta\ d\varphi\ \ dr}$$

The ranges of the co-ordinate variables are

$$\boldsymbol{0} \leq \boldsymbol{r} < \infty$$

$$\boldsymbol{0} \leq \boldsymbol{\theta} < \pi$$

$$\boldsymbol{0} < \varphi < 2\boldsymbol{\pi}$$

Now the unit vector$\boldsymbol{a_r}$, $\boldsymbol{a_\theta}$ and $\boldsymbol{a_\varphi}$ are mutually orthogonal:$\boldsymbol{a_r}$ being directed along the radius or in the direction of increasing $\boldsymbol{r}$, $\boldsymbol{a_\theta}$ in the direction of increasing θ and $\boldsymbol{a_\varphi}$ in the direction of creasing φ.

$$\boldsymbol{a_r}.\,\boldsymbol{a_r} = \boldsymbol{a_\theta}.\,\boldsymbol{a_\theta} = \boldsymbol{a_\varphi}.\,\boldsymbol{a_\varphi} = 1$$

$$\boldsymbol{a_r}.\,\boldsymbol{a_\theta} = \boldsymbol{a_\theta}.\,\boldsymbol{a_\varphi} = \boldsymbol{a_\varphi}.\,\boldsymbol{a_r} = 0$$

$$\boldsymbol{a_r} \times \boldsymbol{a_\theta} = \boldsymbol{a_\varphi}$$

$$\boldsymbol{a_\theta} \times \boldsymbol{a_\varphi} = \boldsymbol{a_r}$$

$$\boldsymbol{a_\varphi} \times \boldsymbol{a_r} = \boldsymbol{a_\theta}$$

TRANSFORMATION OF CO-ORDINATE SYSTEM

It is necessary to transform a vector from one co-ordinate system to another co-ordinate system. Transformation of a vector between Cartesian and cylindrical co-ordinate system and Cartesian and spherical system are carried out.

(A) Transformation between Cartesian and cylindrical systems

A vector in Cartesian co-ordinate system can be converted into cylindrical co-ordinates system.

(i) Conversion of Cartesian to Cylindrical system

The Cartesian co-ordinate system $\boldsymbol{x, y, z}$ can be converted into cylindrical co-ordinates system. $(\boldsymbol{\rho, \varphi, z})$

Given	Transform
$\boldsymbol{x}$	$\boldsymbol{\rho = \sqrt{x^2 + y^2}}$
$\boldsymbol{y}$	$\boldsymbol{\varphi = \tan^{-1}\left(\frac{y}{x}\right)}$
$\boldsymbol{z}$	$\boldsymbol{z = z}$

(ii) Conversion of Cylindrical to Cartesian system

The cylindrical co-ordinates system. $(\boldsymbol{\rho}, \boldsymbol{\varphi}, \boldsymbol{z})$ can be converted into Cartesian co-ordinate system $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$

Given	**Transform**
$\boldsymbol{\rho}$	$\boldsymbol{x = \rho \cos \varphi}$
$\boldsymbol{\varphi}$	$\boldsymbol{y = \rho \sin \varphi}$
$\boldsymbol{z}$	$\boldsymbol{z = z}$

(B) Transformation between Cartesian and Spherical systems

A vector in Cartesian co-ordinate system can be converted into spherical co-ordinates system.

(i) Conversion of Cartesian to Spherical system

The Cartesian co-ordinate system $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ can be converted into cylindrical co-ordinates system. $(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\varphi})$

Given	**Transform**
$\boldsymbol{x}$	$\boldsymbol{r = \sqrt{x^2 + y^2 + z^2}}$
$\boldsymbol{y}$	$\boldsymbol{\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)}$
	$\boldsymbol{= \cos^{-1}\left(\frac{z}{r}\right)}$
$\boldsymbol{z}$	$\boldsymbol{\varphi = \tan^{-1}\left(\frac{y}{x}\right)}$

(ii) Conversion of Spherical to Cartesian co-ordinate system

The Cartesian co-ordinate system $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ can be converted into cylindrical co-ordinates system. $(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\varphi})$

Given	Transform
$\boldsymbol{r}$	$\boldsymbol{x = r \sin\theta \cos\varphi}$
$\boldsymbol{\theta}$	$\boldsymbol{y = r \sin\theta \sin\varphi}$
$\boldsymbol{\varphi}$	$\boldsymbol{z = r \cos\theta}$