3.4 Taylor's Series for Functions of Two Variables

Taylor's expansion for a function of two variables:

Let $f(x,y)$ be a function of two variables x, y. We can expand $f(x+h, y+k)$ in a series of ascending powers of h and k. Consider $f\,(x + h, y + k)$ as a function of the single variable x. i.e., keep y temporarily constant. By Taylor's theorem, We have

$$f\,(x + h, y + k) = f(x, y+k) + \frac{h}{1!}\frac{\partial}{\partial x} f(x, y+k) + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2} f(x, y+k) + \cdots \quad \ldots(1)$$

Now, considering $f\,(x, y + k)$ as a function of y only, we have

$$f\,(x, y + k) = f(x,y) + \frac{k}{1!}\frac{\partial}{\partial y} f(x,y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f(x,y) + \cdots \quad \ldots(2)$$

Differentiating (2) partially with respect to x, we have

$$\frac{\partial}{\partial x} f(x, y+k) = \frac{\partial}{\partial x} f(x,y) + \frac{k}{1!}\frac{\partial^2}{\partial x \partial y} f(x,y) + \frac{k^2}{2!}\frac{\partial^2}{\partial x \partial y^2} f(x,y) + \cdots \quad \ldots(3)$$

Differentiating (3) partially with respect to x, we have

$$\frac{\partial^2}{\partial x^2} f\,(x, y + k) = \frac{\partial^2}{\partial x^2} f(x,y) + \frac{k}{1!}\frac{\partial^3}{\partial x^2 \partial y} f(x,y) + \frac{k^2}{2!}\frac{\partial^4}{\partial x^2 \partial y^2} f(x,y) + \quad \ldots(4)$$

Substituting (2), (3), (4) etc. in (1) we have

$$f\,(x + h, y + k) = f(x,y) + \frac{k}{1!}\frac{\partial}{\partial y} f(x,y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f(x,y) + \cdots$$

$$+ \mathrm{h}\left[\frac{\partial}{\partial x} f(x,y) + \mathrm{k}\,\frac{\partial^2}{\partial x \partial y} f(x,y) + \frac{k^2}{2!}\frac{\partial^3}{\partial x \partial y^2} f(x,y) + \cdots\right]$$

$$+ \frac{h^2}{2!}\left[\frac{\partial^2}{\partial x^2} f(x,y) + \mathrm{k}\,\frac{\partial^3}{\partial x^2 \partial y} f(x,y)\right] + \frac{k^2}{2!}\frac{\partial^4}{\partial x^2 \partial y^2} f(x,y) + \cdots + \cdots$$

$$= f(x,y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \text{higher order items}$$

$$= f(x,y) + \frac{1}{1!}\left[h f_x(x,y) + k f_y(x,y) + \frac{1}{2!}\left[h^2 f_{xx}(x,y) + 2hk f_{xy}(x,y) + k^2 f_{yy}(x,y)\right] + \cdots\right] \quad \ldots(5)$$

The above result can be written in symbolic form as

$$f\ (x\ +h, y\ +k\ ) = f(x,y) + \frac{1}{1!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x,y)$$

$$+\frac{1}{2!}\left(\mathrm{h}\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x,y) + \cdots \qquad \ldots(6)$$

Equation (5) represents an expansion of $f(x\ +\ h,\ y\ +\ k)$ in powers of h and k. From this, we can obtain a form which closely resembles the one dimensional form of Taylor's series.

In (5), replace (x, y) by (a, b)

We *have* $f(a+h, b+k\ )\ =\ f(a,b) + \frac{1}{1!}\left[\left[hf_x(a,b) + kf_y(a,b)\right] + \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hkf_{xy}(a,b) +\ k^2 f_{yy}(a,b)\right] + \cdots\right]$ + higher order terms ... (7)

In equation (7), replace h by $(x\ -\ a)$ and k by $(y\ -\ k)$

We have, then

$$\boldsymbol{f\ (x, y\ ) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] +}$$

$$\boldsymbol{\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots}$$

It is the Taylor's series expansion of $f(x, y)$ about the point (a, b).

Problems Based on Taylor's Series for Function of Two variables

Example:

(i) Expand $e^x cosy$ about $\left(0, \frac{\pi}{2}\right)$up to the third term using Taylor's series.

(ii)$e^x cosy$ in powers of x and y as far as the terms of the third degree .

Solution:

Function	**Value at $\left(0, \frac{\pi}{2}\right)$**	**Value at (0, 0)**
$f\ (x, y\ ) = e^x cosy$	$f = 0$	1

$f_x = e^x cosy$ $f_y = -e^x siny$	$f_x = 0$ $f_y = -1$	1 0
$f_{xx} = e^x cosy$ $f_{xy} = -e^x$siny $f_{yy} = -e^x$cosy	$f_{xx} = 0$ $f_{xy} = -1$ $f_{yy} = 0$	1 0 -1
$f_{xxx} = e^x cosy$ $f_{xxy} = -e^x$siny $f_{xyy} = -e^x cosy$ $f_{yyy} = e^x$sin y	$f_{xxx} = 0$ $f_{xxy} = -1$ $f_{xyy} = 0$ $f_{yyy} = 1$	1 0 -1 0

By Taylor's theorem

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] +$$

$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

$$+\frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] + \cdots$$

(i) $\boldsymbol{a = 0, b = \frac{\pi}{2}}$

$$f(x,y) = 0 + \frac{1}{1!}\left[(x)(0) + \left(y-\frac{\pi}{2}\right)(-1)\right] + \frac{1}{2!}\left[(x)^2(0) + 2(x)(y-\frac{\pi}{2})(-1) + \left(y-\frac{\pi}{2}\right)^2(0)\right]$$

$$+\frac{1}{3!}\left[(x)^3(0) + 3(x)^2\left(y-\frac{\pi}{2}\right)(-1) + 3(x)\left(y-\frac{\pi}{2}\right)^2(0) + \left(y-\frac{\pi}{2}\right)^3(1)\right] + \cdots$$

$$= -y + \frac{\pi}{2} + \frac{1}{2!}\left[-2xy + 2x\frac{\pi}{2}\right] + \frac{1}{3!}\left[-3x^2y + 3\frac{\pi}{2}x^2 + \left(y-\frac{\pi}{2}\right)^3\right]$$

(ii) $\boldsymbol{a = 0, b = 0}$

$$f(x,y) = 1 + \frac{1}{1!}[(x)(1) + (y)(0)] + \frac{1}{2!}[(x)^2(1) + 2(x)(y)(0) + (y)^2(-1)]$$

$$+\frac{1}{3!}[(x)^3(1) + 3(x)^2(y)(0) + 3(x)(y)^2(-1) + (y)^3(0)] + \cdots$$

$$f(x,y) = 1 + x + \frac{1}{2!}[x^2 - y^2] + \frac{1}{3!}[x^3 - 3xy^2] + \cdots$$

Example:

Obtain terms up to the third degree in the Taylor series expansion of $e^x siny$ about the point $\left(1, \frac{\pi}{2}\right)$

Solution:

Function	**Value at $(1, \frac{\pi}{2})$**
$f(x,y) = e^x siny$	$f = e$
$f_x = e^x siny$ $f_y = e^x cosy$	f_x = e $f_y = 0$
$f_{xx} = e^x siny$ $f_{xy} = e^x \cos y$ $f_{yy} = -e^x \sin y$	f_{xx} = e $f_{xy} = 0$ $f_{yy} = -e$
$f_{xxx} = e^x siny$ $f_{xxy} = e^x \cos y$ $f_{xyy} = -e^x siny$ $f_{yyy} = -e^x \cos y$	f_{xxx} = e $f_{xxy} = 0$ $f_{xyy} = -e$ $f_{yyy} = 0$

By Taylor's theorem

$$f(x,y) = f(a,b)+\frac{1}{1!}\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b)+3(x-a)^2(y-b)f_{xxy}(a,b)+3(x-a)(y-b)^2 f_{xyy}(a,b)+(y-b)^3 f_{yyy}(a,b)\right]+\cdots$$

Put $\boldsymbol{a = 1, b = \frac{\pi}{2}}$

$$f(x,y) = e+\frac{1}{1!}\left[(x-1)e+\left(y-\frac{\pi}{2}\right)(0)\right]+$$
$$\frac{1}{2!}\left[(x-1)^2 e+2(x-1)(y-\frac{\pi}{2})(0)+\left(y-\frac{\pi}{2}\right)^2(-e)\right]+$$
$$\frac{1}{3!}\left[(x-1)^3 e+3(x-1)^2\left(y-\frac{\pi}{2}\right)(0)+3(x-1)\left(y-\frac{\pi}{2}\right)^2(-e)+\left(y-\frac{\pi}{2}\right)^3(0)\right]+\cdots$$

$$f(x,y) = e+\frac{1}{1!}(x-1)e+\frac{1}{2!}\left[(x-1)^2 e+\left(y-\frac{\pi}{2}\right)^2(-e)\right]$$
$$+\frac{1}{3!}\left[(x-1)^3 e-3e(x-1)\left(y-\frac{\pi}{2}\right)^2\right]+\cdots$$

Example:

Expand the function $\boldsymbol{\sin xy}$ in powers of $\boldsymbol{x-1}$ and $\boldsymbol{y-\frac{\pi}{2}}$ upto second degree terms.

Solution:

Function	**Value at $(1,\frac{\pi}{2})$**
$f(x,y) = sinxy$	$f = 1$
$f_x = ycos(xy)$ $f_y = xcos(xy)$	$f_x = 0$ $f_y = 0$

$f_{xx} = -y^2\sin(xy)$ $f_{xy} = -xy\,\sin(xy)+\cos(xy)$ $f_{yy} = -x^2\sin xy$	$f_{xx} = -\frac{\pi^2}{4}$ $f_{xy} = -\frac{\pi}{2}$ $f_{yy} = -1$

By Taylor's theorem

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] +$$
$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

Put $a = 1, b = \frac{\pi}{2}$

$$f(x,y) = 1 + \frac{1}{1!}\left[(x-1)(0) + \left(y-\frac{\pi}{2}\right)(0)\right] +$$
$$\frac{1}{2!}\left[(x-1)^2\left(-\frac{\pi^2}{4}\right) + 2(x-1)\left(y-\frac{\pi}{2}\right)\left(-\frac{\pi}{2}\right) + \left(y-\frac{\pi}{2}\right)^2(-1)\right] + \cdots$$
$$= 1 + \frac{1}{2!}\left[(x-1)^2\left(-\frac{\pi^2}{4}\right) + 2(x-1)\left(y-\frac{\pi}{2}\right)\left(-\frac{\pi}{2}\right) - \left(y-\frac{\pi}{2}\right)^2\right] + \cdots$$
$$= 1 + \frac{1}{2!}\left[(x-1)^2\left(-\frac{\pi^2}{4}\right) - \pi(x-1)\left(y-\frac{\pi}{2}\right) - \left(y-\frac{\pi}{2}\right)^2\right] + \cdots$$

Example:

Expand $f(x,y) = e^{xy}$ in Taylors Series at (1, 1) upto second degree.

Solution:

Function	**Value at (1,1)**
$f(x,y) = e^{xy}$	$f = e$
$f_x = y\,e^{xy}$ $f_y = x\,e^{xy}$	$f_x = e$ $f_y = e$

$f_{xx}=y^2e^{xy}$ $f_{xy}= x\ y\ e^{xy}+ e^{xy}$ $f_{yy}= x^2\ e^{xy}$	$f_{xx} = e$ $f_{xy} = e + e = 2e$ $f_{yy} = e$

By Taylor's theorem

$$f\ (x,y) = f(a,b)+\frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] +$$

$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

Put $\boldsymbol{a = 1, b = 1}$

$$f\ (x,y) = e +\frac{1}{1!}[(x-1)e + (y-1)(e)] +$$

$$\frac{1}{2!}[(x-1)^2 e + 2(x-1)(y-1)(2e) + (y-1)^2(e)] + \cdots$$

Example:

Expand $\boldsymbol{e^x \log(1+y)}$ in powers of x and y upto terms of third degree.

Solution:

Function	**Value at (0 ,0)**
$f\ (x,y) =e^x \log(1+y)$	$f = 0$
$f_x= e^x \log(1+y)$ $f_y= e^x \frac{1}{1+y}$	$f_x = 0$ $f_y = 1$
$f_{xx} = e^x \log(1+y)$ $f_{xy} = e^x \frac{1}{1+y}$ $f_{yy} = -e^x \frac{1}{(1+y)^2}$	$f_{xx} = 0$ $f_{xy} = 1$ $f_{yy} = -1$

$f_{xxx} = e^x \log(1+y)$	$f_{xxx} = 0$
$f_{xxy} = e^x \frac{1}{1+y}$	$f_{xxy} = 1$
$f_{xyy} = -e^x \frac{1}{(1+y)^2}$	$f_{xyy} = -1$
$f_{yyy} = 2\, e^x \frac{1}{(1+y)^3}$	$f_{yyy} = 2$

By Taylor's theorem

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] +$$

$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

$$+\frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] + \cdots$$

Put $\boldsymbol{a = 0, b = 0}$

$$f(x,y) = 0 + \frac{1}{1!}[(x)(0) + (y)(1)] + \frac{1}{2!}[(x)^2(0) + 2(x)(y)(1) + (y)^2(-1)]$$

$$+\frac{1}{3!}[(x)^3(0) + 3(x)^2(y)(1) + 3(x)(y)^2(-1) + (y)^3(2)] + \cdots$$

$$= y + \frac{2xy - y^2}{2!} + \frac{3x^2y - 3x\,y^2 + 2y^3}{3!} + \cdots$$

Example:

Expand $\boldsymbol{x^2y + 3y - 2}$ in powers of $(\boldsymbol{x-1})$ and $(\boldsymbol{y+2})$ up to the third degree term

Solution:

Let $f(x,y) = x^2y + 3y - 2$

Function	**Value at (1, -2)**
$f(x,y) = x^2y + 3y - 2$	$f = -10$
$f_x = 2xy$	$f_x = -4$

$f_y = x^2 + 3$	$f_y = 4$
$f_{xx} = 2y$ $f_{xy} = 2x$ $f_{yy} = 0$	$f_{xx} = -4$ $f_{xy} = 2$ $f_{yy} = 0$
$f_{xxx} = 0$ $f_{xxy} = 2$ $f_{xyy} = 0$ $f_{yyy} = 0$	$f_{xxx} = 0$ $f_{xxy} = 2$ $f_{xyy} = 0$ $f_{yyy} = 0$

By Taylor's theorem

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] + \cdots$$

Put $a = 1, b = -2$

$$f(x,y) = -10 + \frac{1}{1!}[(x-1)(-4) + (y+2)(4)] + \frac{1}{2!}[(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y+2)^2(0)] + \frac{1}{3!}[(x-1)^3(0) + 3(x-1)^2(y+2)(2) + 3(x-1)(y+2)^2(0) + (y+2)^3(0)] + \cdots$$

$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2)$$

Example: Find the Taylor series expansions of $e^x siny$ at the point $\left(-1, \frac{\pi}{4}\right)$ upto third degree terms.

Solution:

Function	Value at $\left(-1, \frac{\pi}{4}\right)$
$f(x, y) = e^x siny$	$f = \frac{1}{e}\frac{1}{\sqrt{2}}$
$f_x = e^x siny$ $f_y = e^x cosy$	$f_x = \frac{1}{e}\frac{1}{\sqrt{2}}$ $f_y = \frac{1}{e}\frac{1}{\sqrt{2}}$
$f_{xx}=e^x siny$ $f_{xy}= e^x \text{cosy}$ $f_{yy}=-e^x \text{siny}$	$f_{xx} = \frac{1}{e}\frac{1}{\sqrt{2}}$ $f_{xy} = \frac{1}{e}\frac{1}{\sqrt{2}}$ $f_{yy} = -\frac{1}{e}\frac{1}{\sqrt{2}}$
$f_{xxx}=e^x siny$ $f_{xxy}=e^x \cos y$ $f_{xyy} = -e^x siny$ $f_{yyy} = -e^x \text{cosy}$	$f_{xxx} = \frac{1}{e}\frac{1}{\sqrt{2}}$ $f_{xxy} = \frac{1}{e}\frac{1}{\sqrt{2}}$ $f_{xyy} = -\frac{1}{e}\frac{1}{\sqrt{2}}$ $f_{yyy} = -\frac{1}{e}\frac{1}{\sqrt{2}}$

By Taylor's theorem

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] +$$

$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

$$+\frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] + \cdots$$

Put $a = -1, b = \frac{\pi}{4}$

$$f(x,y) = \frac{1}{e}\frac{1}{\sqrt{2}} + \frac{1}{1!}\left[(x+1)\frac{1}{e}\frac{1}{\sqrt{2}} + \left(y-\frac{\pi}{4}\right)\frac{1}{e}\frac{1}{\sqrt{2}}\right] +$$

$$\frac{1}{2!}\left[(x+1)^2\frac{1}{e}\frac{1}{\sqrt{2}} + 2(x+1)\left(y-\frac{\pi}{4}\right)\frac{1}{e}\frac{1}{\sqrt{2}} + \left(y-\frac{\pi}{4}\right)^2\frac{1}{e}\frac{1}{\sqrt{2}}\right] +$$

$$\frac{1}{3!}\left[(x+1)^3\frac{1}{e}\frac{1}{\sqrt{2}} + 3(x+1)^2\left(y-\frac{\pi}{4}\right)\frac{1}{e}\frac{1}{\sqrt{2}} + 3(x+1)\left(y-\frac{\pi}{4}\right)^2\left(-\frac{1}{e}\frac{1}{\sqrt{2}}\right) + \left(y-\frac{\pi}{4}\right)^3\left(-\frac{1}{e}\frac{1}{\sqrt{2}}\right)\right] + \cdots$$

Exercise:

1. Use Taylor's formula to expand the function f defined by $f(x,y) = x^2 + xy + y^2$ in powers of $(x-1)$ and $(y-2)$. **[A. U. Tvli. Jan. 2011]**

[**Ans:** $(x-1)^2 + (y-2)^2 + (x-1)(y-2) + 4(x-1) + 5(y-2) + 7$]

2. Expand $f(x,y) = x^y$ in Taylors Series at (1, 1) up to first degree. **[A U, Jan. 2014]**

[**Ans:** $1 + (x-1) + \cdots$]

3. Expand $x^2y^2 + 2x^2y + 3xy^2$ in powers of $(x+2)$ and $(y-1)$ up to the third degree terms.

[A.U A/M 2014) (A.U Jan. 2012]

[**Ans:** $6 + [(x+2)(-9) + (y-1)(4)] + \frac{1}{2!}[(x+2)^2(6) - 20(x+2)(y-1) + (y-1)^2(-4)] + \frac{1}{3!}[(x+2)^2(y-1)(24) + (x+2)(y-1)^2(6)] + \cdots$

4. Expand $e^x \sin y$ in powers of x and y upto terms of third degree. **[A. U M/J 2013]**

[**Ans** : $y + xy$]

5. Expand $f(x, y) = tan^{-1}\left({}^{y}/_{x}\right)$ in powers of (x-1) and (y-1) up to third degree terms

[**Ans:** $f(x, y) = \frac{\pi}{4} - \frac{1}{2}[(x-1) - (y-1)] + \frac{1}{4}\ [(x-1)^2 - (y-1)^2]\ - \frac{1}{12}[(x-1)^3 + \ \ 3(x-1)^2(y-1) - 3(x-1)(y-1)^2 - (y-1)^3] + \cdots$