3.3 Jacobians

If u and v are the functions of two independent variables x and y then

$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of u, v with respect to x and y. it is denoted by $\frac{\partial(u,v)}{\partial(x,y)} = J$

Note:

1. $J' = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$
2. The Jacobian of u, v, w with respect to x, y, z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties of Jacobian:

Property:1

If u and v are the functions of x and y then $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$

Proof:

Let $J = \frac{\partial(u,v)}{\partial(x,y)}$, $J' = \frac{\partial(x,y)}{\partial(u,v)}$

$$J\,J' = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \end{vmatrix} \quad \dots (1)$$

$$\frac{\partial u}{\partial u} = 1 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial u}{\partial v} = 0 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial v}{\partial v} = 1 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial v}{\partial u} = 0 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u}$$

$$(1) \Rightarrow \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\therefore \; J J' = 1$$

Problems based on 1st property:

Example:

If $x = u(1-v)$, $y = uv$ find J and J' and prove that $JJ' = 1$

Solution:

To prove $JJ' = 1$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \ldots (1)$$

Given $x = u(1-v)$, $y = uv$

$$\frac{\partial x}{\partial u} = 1 - v \, , \frac{\partial y}{\partial u} = v$$

$$\frac{\partial x}{\partial v} = -u \quad , \quad \frac{\partial x}{\partial u} = u$$

$$(1) \Rightarrow \quad J = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= u - uv + uv \quad = u$$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \ldots (2)$$

Given $x = u - uv$, $y = uv$

$$x = u - y \, , \; v = \frac{y}{u}$$

$$u = x + y \, , \; v = \frac{y}{x+y}$$

$$\frac{\partial u}{\partial x} = 1 \, , \frac{\partial v}{\partial x} = y\left(-\frac{1}{(x+y)^2}\right) \times 1 = -\frac{y}{(x+y)^2}$$

$$\frac{\partial u}{\partial y} = 1 \, , \frac{\partial v}{\partial y} = \frac{(x+y)\times 1 - y \times 1}{(x+y)^2} = \frac{x}{(x+y)^2}$$

$$(2) \Rightarrow \quad J' = \begin{vmatrix} 1 & 1 \\ -\frac{y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2}$$

$$= \frac{1}{x+y}$$

$$= \frac{1}{u}$$

here $J = u$ and $J' = \frac{1}{u}$

$$\therefore \; JJ' = u \times \frac{1}{u} = 1$$

Hence proved.

Example:

If $x = uv, y = \frac{u}{v}$ prove $\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = 1$

Solution:

Let $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ ... (1)

Given $x = uv, y = \frac{u}{v}$

$$\frac{\partial x}{\partial u} = v, \frac{\partial y}{\partial u} = \frac{1}{v}$$

$$\frac{\partial x}{\partial v} = u \quad , \quad \frac{\partial x}{\partial u} = \frac{-u}{v^2}$$

$$(1) \Rightarrow \quad J = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix}$$

$$= \frac{-uv}{v^2} - \frac{u}{v} = \frac{-2u}{v}$$

$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ ... (2)

$$x = uv, \; y = \frac{u}{v} \; \Rightarrow \; u = vy$$

$$x = vyv \; \Rightarrow x = v^2 y \; \Rightarrow v^2 = \frac{x}{y} \; \Rightarrow v = \frac{\sqrt{x}}{\sqrt{y}}$$

$$x = uv \Rightarrow u = \frac{x}{v} \Rightarrow u = \frac{x\sqrt{y}}{\sqrt{x}} = \sqrt{x}\sqrt{y}$$

$$Now \Rightarrow u = \sqrt{x}\sqrt{y}, \; v = \frac{\sqrt{x}}{\sqrt{y}}$$

$$\frac{\partial u}{\partial x} = \frac{\sqrt{y}}{2\sqrt{x}}, \frac{\partial v}{\partial x} = \frac{1}{\sqrt{y}}\frac{1}{2\sqrt{x}}$$

$$\frac{\partial u}{\partial y} = \frac{\sqrt{x}}{2\sqrt{y}}, \frac{\partial v}{\partial y} = \sqrt{x}\left(\frac{-1}{y}\right)\frac{1}{2\sqrt{y}}$$

$$(2) \Rightarrow J' = \begin{vmatrix} \frac{\sqrt{y}}{2\sqrt{x}} & \frac{\sqrt{x}}{2\sqrt{y}} \\ \frac{1}{2\sqrt{y}\sqrt{x}} & \frac{-\sqrt{x}}{2y\sqrt{y}} \end{vmatrix}$$

$$= -\frac{1}{4y} - \frac{1}{4y} = -\frac{2}{4y} = -\frac{1}{2y}$$

$$= -\frac{v}{2u} \qquad (\because y = \frac{u}{v})$$

here $J = \frac{-2u}{v}$ and $J' = -\frac{v}{2u}$

$$\therefore \; JJ' = \frac{-2u}{v} \times \left(-\frac{v}{2u}\right) = 1$$

Hence proved.

Example:

If $x = v^2+w^2, y = w^2+u^2, z = u^2+v^2$ find J and J' [AU Apr2005]

Solution:

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \dots (1)$$

Given $x = v^2 + w^2, y = w^2 + u^2, z = u^2 + v^2$

$$\frac{\partial x}{\partial u} = 0, \quad \frac{\partial y}{\partial u} = 2u, \quad \frac{\partial z}{\partial u} = 2u$$

$$\frac{\partial x}{\partial v} = 2v, \quad \frac{\partial y}{\partial v} = 0, \quad \frac{\partial z}{\partial v} = 2v$$

$$\frac{\partial x}{\partial w} = 2w, \quad \frac{\partial y}{\partial w} = 2w, \quad \frac{\partial z}{\partial w} = 0$$

$$(1) \Rightarrow J = \begin{vmatrix} 0 & 2v & 2w \\ 2u & 0 & 2w \\ 2u & 2v & 0 \end{vmatrix} = 0(0 - 4vw) - 2v(0 - 4uw) + 2w(4uv)$$

$$= 8vwu + 8wuv$$

$$J = 16uvw$$

By property 1 $\quad J\,J' = 1 \Rightarrow J' = \frac{1}{J} = \frac{1}{16uvw}$

Property: 2

If u and v are the functions of r , s where r , s are the functions of x and y then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$

Proof:

$$\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \times \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \frac{\partial(u,v)}{\partial(x,y)}$$

Similarly $\quad \frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\partial(u,v,w)}{\partial(r,s,t)} \times \frac{\partial(r,s,t)}{\partial(x,y,z)}$

Problems based on 2nd property:

Example:

If $u = 2xy$, $v = x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$. Evaluate $\frac{\partial(u,v)}{\partial(r,\theta)}$ without actual substitution.

Solution:

By property: 2

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \ldots (1)$$

Given $u = 2xy$, $v = x^2 - y^2$, $x = rcos\theta$, $y = rsin\theta$

$$\frac{\partial u}{\partial x} = 2y, \frac{\partial v}{\partial x} = 2x, \quad \frac{\partial x}{\partial r} = cos\theta, \frac{\partial y}{\partial r} = sin\theta$$

$$\frac{\partial u}{\partial y} = 2x, \frac{\partial v}{\partial y} = -2y, \quad \frac{\partial x}{\partial \theta} = -rsin\theta, \frac{\partial y}{\partial \theta} = rcos\theta$$

$$(1) \Rightarrow \frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \times \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix}$$

$$= (-4y^2 - x^2)(rcos^2\theta + rsin^2\theta)$$

$$= -4(x^2 + y^2) \times r \qquad \left(\because \begin{array}{l} x^2 = r^2cos^2\theta \\ y^2 = r^2sin^2\theta \end{array}\right)$$

$$= -4r^2 \times r$$

$$= -4r^3$$

Example:

If $x = a(u+v)$, $y = b(u-v)$ and $u = r^2cos2\theta$, $v = r^2sin2\theta$ then evaluate $\frac{\partial(x,y)}{\partial(r,\theta)}$

Solution:

By Property: 2

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \times \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} \ldots (1)$$

Given $x = a(u+v)$, $y = b(u-v)$, $u = r^2cos2\theta$, $v = r^2sin2\theta$

$$\frac{\partial x}{\partial u} = a, \frac{\partial y}{\partial u} = b, \quad \frac{\partial u}{\partial r} = 2rcos2\theta, \frac{\partial v}{\partial r} = 2rsin\theta$$

$$\frac{\partial x}{\partial v} = a, \frac{\partial y}{\partial v} = -b, \quad \frac{\partial u}{\partial \theta} = -2r^2sin2\theta, \frac{\partial v}{\partial \theta} = 2r^2cos2\theta$$

$$(1) \Rightarrow = \begin{vmatrix} a & a \\ b & -b \end{vmatrix} \times \begin{vmatrix} 2rcos2\theta & -2r^2sin2\theta \\ 2rsin2\theta & 2r^2cos\theta \end{vmatrix}$$

$$= (-ab - ab)(4r^3cos^2 2\theta + 4r^3sin^2 2\theta)$$

$$= -2ab \times 4r^3$$
$$= -8r^3ab$$

Property: 3

$If\ u, v, w$ are functionally dependent of three independent variables x, y, z then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

Proof:

As u, v, w are not independent then $f(u, v, w) = 0 \ldots (1)$

Differentiating equation (1) with respect to x, y, z we get

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = 0 \ \ldots (2)$$

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} = 0 \ \ldots (3)$$

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} = 0 \ \ldots (4)$$

Eliminating f derivatives from (2) ,(3) and (4) we have

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{vmatrix} = 0$$

On interchanging rows and columns, we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = 0$$

i.e) $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

Example:

If $p = 3x + 2y - z, q = x - 2y = z, r = x + 2y - z$ prove that p, q, r are functionally dependent.

Solution:

To prove p, q, r are functionally dependent.

i.e) To prove $\frac{\partial(p,q,r)}{\partial(x,y,z)} = 0$

$$\text{Consider } \frac{\partial(p,q,r)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} & \frac{\partial p}{\partial z} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} & \frac{\partial q}{\partial z} \\ \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 2 & -1 \\ 1 & -2 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

= 3(2-2) – 2(-1 -1) -1(2+2)

= 4 – 4

= 0

$\therefore p , q , r$ are functionally dependent.

Example:

Prove that $u = x + y + z$, $v = xy + yz + zx$, $w = x^2 + y^2 + z^2$ are functionally dependent. Find the relation between them.

Solution:

To prove u , v , w are functionally dependent

i.e) To prove $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

$$\text{Consider } \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ y+z & z+x & x+y \\ 2x & 2y & 2z \end{vmatrix}$$

$$= 2x^2 + 2zx - 2xy - 2y^2 - 2zy - 2z^2 + 2xy$$
$$+ 2x^2 + 2y^2 + 2yz - 2xz - 2x^2$$

= 0

$\therefore$ Given u , v , w are functionally dependent

The relation between them is $(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx)$

$$u^2 = w + 2v$$

Example:

If $x+y+z=u, y+z=uv, z=uvw$ prove that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$

Solution:

Given $x+y+z=u,\ y+z=uv,\ z=uvw$

$x+uv=u,\ y+uvw=uv,\ z=uvw$

$x=u-uv,\ y=uv-uvw,\ z=uvw$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial z} \end{vmatrix} \dots (1)$$

$x=u-uv,\ y=uv-uvw,\ z=uvw$

$\frac{\partial x}{\partial u} = 1-v,\quad \frac{\partial y}{\partial u} = v-vw,\quad \frac{\partial z}{\partial u} = vw$

$\frac{\partial x}{\partial v} = -u,\quad \frac{\partial y}{\partial v} = u-uw,\quad \frac{\partial z}{\partial v} = uw$

$\frac{\partial x}{\partial w} = 0,\ \frac{\partial y}{\partial w} = -uv,\ \frac{\partial z}{\partial w} = uv$

$$(1) \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)(u^2v - u^2vw + u^2vw) + u(uv^2 - uv^2w + uv^2w) +$$

$$0(uvw - uvw^2 - uvw + uvw^2$$

$$= (1-v)u^2v + u^2v^2$$

$$= u^2v - u^2v^2 + u^2v^2$$

$$= u^2v$$

Hence proved.

Example:

Find the Jacobian $\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}$ of the transformation $x = rsin\theta cos\varphi$, $y = rsin\theta sin\varphi$, $z = r\cos\theta$

Solution:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} \ldots (1)$$

$$x = rsin\theta cos\varphi , y = rsin\theta sin\varphi , z = rcos\theta$$

$$\frac{\partial x}{\partial r} = \sin\theta cos\varphi , \quad \frac{\partial y}{\partial r} = \sin\theta sin\varphi , \quad \frac{\partial z}{\partial r} = \cos\theta$$

$$\frac{\partial x}{\partial \theta} = r\cos\theta cos\varphi , \quad \frac{\partial y}{\partial \theta} = r\cos\theta sin\varphi , \quad \frac{\partial z}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial x}{\partial \varphi} = -r\sin\theta sin\varphi , \quad \frac{\partial y}{\partial \varphi} = r\sin\theta cos\varphi, \quad \frac{\partial z}{\partial \varphi} = 0$$

$$(1) \Rightarrow \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} \sin\theta cos\varphi & r\cos\theta cos\varphi & -r\sin\theta sin\varphi \\ \sin\theta sin\varphi & r\cos\theta sin\varphi & r\sin\theta cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= sin\theta cos\varphi(0 + r^2 sin^2\theta cos\varphi) -$$

$$rcos\theta cos\varphi(-rsin\theta cos\theta cos\varphi) - r(sin\theta sin\varphi(-rsin^2\theta sin\varphi - rcos^2\theta sin\varphi)$$

$$= r^2 sin^3\theta cos^2\varphi + r^2 cos^2\theta cos^2\varphi sin\theta + r^2 sin^3\theta sin^2\varphi +$$

$$r^2 cos^2\theta sin^2\varphi sin\theta$$

$$= r^2 sin^3\theta + r^2 cos^2\theta sin\theta$$

$$= r^2 sin\theta(sin^2\theta + cos^2\theta)$$

$$= r^2 sin\theta$$

Example:

If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$ show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$

Solution:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \ldots (1)$$

$$u = \frac{yz}{x}, v = \frac{zx}{y}, w = \frac{xy}{z}$$

$$\frac{\partial u}{\partial x} = -\frac{yz}{x^2}, \quad \frac{\partial v}{\partial x} = \frac{z}{y}, \quad \frac{\partial w}{\partial x} = \frac{y}{z}$$

$$\frac{\partial u}{\partial y} = \frac{z}{x}, \quad \frac{\partial v}{\partial y} = -\frac{zx}{y^2}, \quad \frac{\partial w}{\partial y} = \frac{x}{z}$$

$$\frac{\partial u}{\partial z} = \frac{y}{z}, \quad \frac{\partial v}{\partial z} = \frac{x}{y}, \quad \frac{\partial w}{\partial z} = -\frac{xy}{z^2}$$

$$(1) \Rightarrow \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= -\frac{yz}{x^2}\left[\frac{x^2yz}{y^2z^2} - \frac{x^2}{yz}\right] - \frac{z}{x}\left[-\frac{xyz}{yz^2} - \frac{xy}{yz}\right] + \frac{y}{x}\left[\frac{zx}{yz} + \frac{zxy}{zy^2}\right]$$

$$= -\frac{x^2y^2z^2}{x^2y^2z^2} + \frac{yzx^2}{x^2yz} + \frac{xyz^2}{xyz^2} + \frac{xyz}{xyz} + \frac{yzx}{xyz} + \frac{zxy^2}{xzy^2}$$

$$= -1 + 1 + 1 + 1 + 1 + 1 = 4$$

Hence proved.

Example:

Find the value of $\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_2)}$ **if** $\boldsymbol{y_1 = 1 - x_1, y_2 = x_1(1 - x_2), y_3 = x_1x_2(1 - x_3)}$

Solution:

$$\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_2)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_2} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} \ldots (1)$$

Given $y_1 = 1 - x_1$, $y_2 = x_1(1 - x_2), y_3 = x_1x_2(1 - x_3)$

$\frac{\partial y_1}{\partial x_1} = -1$, $\frac{\partial y_2}{\partial x_1} = 1 - x_2$, $\frac{\partial y_3}{\partial x_1} = x_2 - x_2x_3$

$\frac{\partial y_1}{\partial x_2} = 0$, $\frac{\partial y_2}{\partial x_2} = -x_1$, $\frac{\partial y_3}{\partial x_2} = x_1 - x_1x_3$

$\frac{\partial y_1}{\partial 3} = 0$, $\frac{\partial y_2}{\partial x_3} = 0$, $\frac{\partial y_3}{\partial x_3} = 0 - x_1x_2$

$$(1) \Rightarrow \frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_2)} = \begin{vmatrix} -1 & 0 & 0 \\ 1 - x_2 & -x_1 & 0 \\ x_2 - x_2x_3 & x_1 - x_1x_3 & -x_1x_2 \end{vmatrix}$$

$$= -1(x_1^2x_2 - 0) = -x_1^2x_2$$

Example:

If $x = acosh\varphi cos\theta$, $y = asinh\varphi sin\theta$ show that

$$\frac{\partial(x,y)}{\partial(\varphi,\theta)} = \frac{a^2}{2}(cosh2\varphi - cos2\theta)$$

Solution:

$$\frac{\partial(x,y)}{\partial(\varphi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \end{vmatrix} \dots (1)$$

Given $x = acosh\varphi cos\theta$, $y = asinh\varphi sin\theta$

$\frac{\partial x}{\partial \varphi} = a\cos\theta sinh\varphi$, $\frac{\partial y}{\partial \varphi} = a\sin\theta cosh\varphi$

$\frac{\partial x}{\partial \theta} = -a\sin\theta cosh\varphi$, $\frac{\partial y}{\partial \theta} = a\cos\theta sinh\varphi$

$$(1) \Rightarrow \frac{\partial(x,y)}{\partial(\varphi,\theta)} = \begin{vmatrix} a\cos\theta sinh\varphi & -a\sin\theta cosh\varphi \\ a\sin\theta cosh\varphi & a\cos\theta sinh\varphi \end{vmatrix}$$

$$= a^2sinh^2\varphi cos^2\theta + a^2cosh^2\varphi sin^2\theta$$

$$= a^2[sinh^2\varphi(1 - sin^2\theta) + (1 + sinh^2\varphi)sin^2\theta]$$

$$= a^2[sinh^2\varphi - sinh^2\varphi sin^2\theta + sin^2\theta + sinh^2\varphi sin^2\theta]$$

$$= a^2[sinh^2\varphi + sin^2\theta]$$

$$= a^2[\frac{cosh2\varphi - 1}{2} + \frac{1 - cos2\theta}{2}]$$

$$= a^2[\frac{cosh2\varphi - 1 + 1 - cos2\theta}{2}]$$

$$= \frac{a^2}{2}(\cos 2\varphi - cos 2\theta)$$

Hence proved.

Jacobian of Implicit Functions

If $f_1(u, v, w, x, y, z) = 0$

$f_2(u, v, w, x, y, z) = 0$

and $f_3(u, v, w, x, y, z) = 0$ are three implicit functions, we can consider u, v, w as implicit function x, y, z then it can be proved

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

Also $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \frac{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}$$

Example:

If $f_1 = u - x - y - z = 0$, $f_2 = uv - y - z = 0$, $f_3 = uvw - z = 0$, prove that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$

Solution:

Given $f_1 = u - x - y - z$, $f_2 = uv - y - z$, $f_3 = uvw - z$

$$\frac{\partial f_1}{\partial x} = -1 , \frac{\partial f_1}{\partial u} = 1 , \quad \frac{\partial f_2}{\partial x} = 0 , \frac{\partial f_2}{\partial u} = v , \quad \frac{\partial f_3}{\partial x} = 0 , \frac{\partial f_3}{\partial u} = vw$$

$$\frac{\partial f_1}{\partial y} = -1 , \frac{\partial f_1}{\partial v} = 0 , \frac{\partial f_2}{\partial y} = -1 , \frac{\partial f_2}{\partial v} = u , \quad \frac{\partial f_3}{\partial y} = 0 , \frac{\partial f_3}{\partial v} = uw$$

$$\frac{\partial f_1}{\partial z} = -1 , \frac{\partial f_1}{\partial w} = 0 , \frac{\partial f_2}{\partial z} = -1 , \frac{\partial f_2}{\partial w} = 0 , \quad \frac{\partial f_3}{\partial z} = -1 , \frac{\partial f_3}{\partial w} = uv$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \frac{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}$$

$$= -\frac{\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial 3}{\partial w} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}} = -\frac{\begin{vmatrix} 1 & 0 & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix}}{\begin{vmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix}} = -\frac{u^2v}{-1\times 1}$$

$= u^2v$

Hence proved.

Exercise:

1. If $u = \frac{y^2}{x}$, $v = \frac{x^2}{y}$ find $\frac{\partial(u,v)}{\partial(x,y)}$ **Ans:** -3

2. If $u = xy$, $v = x + y$ find $\frac{\partial(x,y)}{\partial(u,v)}$ **Ans:** $\frac{1}{y-x}$

3. If $u = x + y + z$, $u^2v = y + z, u^3w = z$ show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = u^5$

4. Find the Jacobian of $x = 2u,\ y = 3v^2, z = 4w^3$ **Ans:** $144uw^2$

5. If $x = e^r\sec\theta$, $y = e^r\tan\theta$, prove $\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

6. If $x = 4(1-v)$, $y = uv$ compute J $= \frac{\partial(x,y)}{\partial(u,v)}$ and J ' $= \frac{\partial(u,v)}{\partial(x,y)}$ and verify JJ ' = 1

7. If $y_1 = \frac{x_2x_3}{x_1}$, $y_2 = \frac{x_1x_3}{x_2}$, $y_3 = \frac{x_1x_2}{x_3}$ find $\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)}$ **Ans:**4

8. If $x = u(1 + v)$, $y = v(1 + v)$ find $\frac{\partial(x,y)}{\partial(u,v)}$ **Ans:** $-\frac{y}{2x}$

9. If $x = r\cos\theta, y = r\sin\theta$ find (i) $\frac{\partial(x,y)}{\partial(r,\theta)}$ **Ans:** r

   (ii) $\frac{\partial(r,\theta)}{\partial(x,y)}$ **Ans:** $\frac{1}{r}$

10. Show that the function $u = x + y - z$, $v = x - y + z$, $w = x^2 + y^2 + z^2 - 2yz$ are dependent. Find the relation between them. **Ans:** $u^2+v^2 = 2w$

11. Are $u = \frac{x}{y}$, $v = \frac{x+y}{x-y}$ functionally dependent? If so, Find the relation between them.

. **Ans:** Yes, $(1-u)(1-v) = 0$

12. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = x^3 + y^3 + z^3 - 3xyz$,prove that u, v, w are not dependent and find their relation. **Ans:** $2w = u(3v - u^2)$

13. Examine the functional dependence of the functions $u = \frac{x+y}{x-y}$, $v = \frac{xy}{(x-y)^2}$? If so, Find the relation between them. **Ans:** $u^2 - 4v = 1$