5.5 Z-Transforms and its Properties

1. **Find** $Z[\cos n\theta]$ **,** $Z[\sin n\theta]$ **and hence find i)** $Z\left[\cos\frac{n\pi}{2}\right]$, ii) $Z\left[\sin\frac{n\pi}{2}\right]$

iii) $Z\left[r^n\cos n\theta\right]$ iv) $Z\left[r^n\sin n\theta\right]$

Solution:

We know that $e^{in\theta} = \cos n\theta + \sin n\theta$

$\cos n\theta =$ real part of $e^{in\theta}$ & $\sin n\theta =$ imaginary part of $e^{in\theta}$

and $Z\left[a^n\right] = \frac{z}{z-a}$

$$Z\left[e^{in\theta}\right] = Z\left[\left(e^{i\theta}\right)^n\right] = \frac{z}{z-e^{i\theta}}$$

$$= \frac{z}{z-(\cos\theta + i\sin\theta)}$$

$$= \frac{z}{(z-\cos\theta)-i\sin\theta}\times\frac{(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)+i\sin\theta}$$

$$Z\left[e^{in\theta}\right] = \frac{z(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)^2-i^2\sin^2\theta} \quad \because (a+b)(a-b)=a^2-b^2$$

$$Z\left[\cos n\theta + i\sin n\theta\right] = \frac{z(z-\cos\theta)+iz\sin\theta}{z^2-2z\cos\theta+\cos^2\theta+\sin^2\theta} \quad \because i^2=-1$$

$$Z\left[\cos n\theta\right] + iZ\left[\sin n\theta\right] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} + i\frac{z\sin\theta}{z^2-2z\cos\theta+1} \quad \because \cos^2\theta+\sin^2\theta=1$$

Equating co-efft. Of real and img parts on both sides

$$Z\left[\cos n\theta\right] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} \quad ; \; Z\left[\sin n\theta\right] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

Deduction:

We know that

$$Z\left[\cos n\theta\right] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$$

i) $$Z\left[\cos\frac{n\pi}{2}\right] = Z\left[\cos n\theta\right]_{\theta\to\frac{\pi}{2}} = \frac{z\left(z-\cos\frac{\pi}{2}\right)}{z^2-2z\cos\frac{\pi}{2}+1}$$

$$Z\left[\cos\frac{n\pi}{2}\right] = \frac{z^2}{z^2+1} \quad \because \cos\frac{\pi}{2}=0$$

$$Z\left[\sin n\theta\right] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

ii) $$Z\left[\sin\frac{n\pi}{2}\right] = Z\left[\sin n\theta\right]_{\theta\to\frac{\pi}{2}} = \frac{z\sin\frac{\pi}{2}}{z^2-2z\cos\frac{\pi}{2}+1}$$

$$\therefore Z\left[\sin\frac{n\pi}{2}\right] = \frac{z}{z^2+1} \quad \because \cos\frac{\pi}{2}=0 \;\&\; \sin\frac{\pi}{2}=1$$

We know that

$$Z\left[a^n f(n)\right] = Z\left[f(n)\right]_{z\to\frac{z}{a}}$$

iii) $Z\left[r^n \cos n\theta\right] = \left[Z\left[\cos n\theta\right]\right]_{z\to\frac{z}{r}}$

$$= \left[\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}\right]_{z\to\frac{z}{r}}$$

$$= \left[\frac{\frac{z}{r}\left(\frac{z}{r}-\cos\theta\right)}{{\frac{z}{r^2}}^{2}-\frac{2z}{r}\cos\theta+1}\right]$$

$$= \frac{\frac{z}{r}\left(\frac{z-r\cos\theta}{r}\right)}{\frac{z^2-2zr\cos\theta+r^2}{r^2}}$$

$$Z\left[r^n \cos n\theta\right] = \frac{z(z-r\cos\theta)}{z^2-2zr\cos\theta+r^2}$$

iv) $Z\left[r^n \sin n\theta\right] = \left[Z\left\{\sin n\theta\right\}\right]_{z\to\frac{z}{r}} = \frac{\frac{z}{r}\sin\theta}{\frac{z^2}{r^2}-2\frac{z}{r}\cos\theta+r^2} = \frac{\frac{z}{r}\sin\theta}{\frac{z^2-2zr\cos\theta+r^2}{r^2}}$

$$Z\left[r^n \sin n\theta\right] = \frac{zr\sin\theta}{z^2-2zr\cos\theta+r^2}$$

2. **Find the Z-transform of** $\frac{1}{n(n+1)}$, $for\ n\ge 1$

Solution

$$Z\left[\frac{1}{n(n+1)}\right] = ?$$

By partial Fraction:

$$\frac{1}{n(n+1)} = \frac{A}{n}+\frac{B}{n+1}$$

$$1 = A(n+1)+Bn$$

Put $n=-1$; $1=-B \Rightarrow B=-1$

Put $n=0$; $A=1$

$$\frac{1}{n(n+1)} = \frac{1}{n}-\frac{1}{n+1}$$

$$Z\left[\frac{1}{n(n+1)}\right] = Z\left[\frac{1}{n}\right]-Z\left[\frac{1}{n+1}\right] ------(1)$$

Now, we know that

$$Z\left[f(n)\right] = \sum_{n=0}^{\infty} f(n)z^{-n}$$

$$Z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{z}\right)^n \quad \because n>0$$

$$= \frac{1}{z}+\frac{1}{2}\left(\frac{1}{z}\right)^2+\frac{1}{3}\left(\frac{1}{z}\right)^3+....$$

	$= x + \frac{x^2}{2} + \frac{x^3}{3} + ...$ here $\frac{1}{z} = x$ $= -\log(1-x)$ $Z\left[\frac{1}{n}\right] = -log\left(1-\frac{1}{z}\right) = -\log\left(\frac{z-1}{z}\right) = \log\left(\frac{z}{z-1}\right)$ $Z\left[\frac{1}{n}\right] = \log\left(\frac{z}{z-1}\right)$ $Z\left(\frac{1}{n+1}\right) = \sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{1}{z}\right)^n$ $= 1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + ...$ $= z\left[\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + ..\right]$ $= z\left[-\log\left(1-\frac{1}{z}\right)\right] = -z\log\left(\frac{z-1}{z}\right)$ $Z\left(\frac{1}{n+1}\right) = z\log\left(\frac{z}{z-1}\right)$ $(1) \Rightarrow \quad Z\left[\frac{1}{n(n+1)}\right] = \log\left(\frac{z}{z-1}\right) + z\log\left(\frac{z}{z-1}\right)$ $\therefore Z\left[\frac{1}{n(n+1)}\right] = (z+1)\log\left(\frac{z}{z-1}\right)$
3.	**Find** $Z\left[n(n-1)(n-2)\right]$ **.** **Solution:** $Z\left[n(n-1)(n-2)\right] = Z\left[(n^2-n)(n-2)\right] = Z\left[n^3 - 2n^2 - n^2 + 2n\right] = Z\left[n^3 - 3n^2 + 2n\right]$ $Z\left[n(n-1)(n-2)\right] = Z\left[n^3\right] - 3Z\left[n^2\right] + 2Z\left[n\right] \quad ----(1)$ We know that $Z\left[f(n)\right] = \sum_{n=0}^{\infty} f(n)z^{-n}$ $Z\left[n\right] = \sum_{n=0}^{\infty} n\left(\frac{1}{z}\right)^n$ $= 0 + 1\left(\frac{1}{z}\right)^1 + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right)^3 + ...$ $= x + 2x^2 + 3x^3 + ...$ $= x(1 + 2x + 3x^2 + ...) = x(1-x)^{-2} = \frac{1}{z}\left(1-\frac{1}{z}\right)^{-2}$ $= \frac{1}{z}\left(\frac{z-1}{z}\right)^{-2} = \frac{1}{z}\left(\frac{z}{z-1}\right)^2 = \frac{1}{z}\left(\frac{z^2}{(z-1)^2}\right)$ $Z\left[n\right] = \frac{z}{(z-1)^2}$ We know that $Z\left[nf(n)\right] = -z\frac{d}{dz}\left\{Z\left[f(n)\right]\right\}$

	$Z\left[n^2\right] = -z\frac{d}{dz}\{Z[n]\}$ $= -z\frac{d}{dz}\left\{\frac{z}{(z-1)^2}\right\}$ $= -z\left\{\frac{(z-1)^2(1) - z[2(z-1)]}{(z-1)^4}\right\}$ $= -z\left\{\frac{(z-1)(z-1-2z)}{(z-1)^4}\right\}$ $= -z\left\{\frac{-1-z}{(z-1)^3}\right\}$ $Z\left[n^2\right] = \frac{z+z^2}{(z-1)^3}$ $Z\left[n^3\right] = Z\left[n n^2\right] = -z\frac{d}{dz}\left\{Z\left[n^2\right]\right\}$ $= -z\frac{d}{dz}\left\{\frac{z+z^2}{(z-1)^3}\right\}$ $= -z\left[\frac{(z-1)^3(2z+1) - (z^2+z)3(z-1)^2(1-0)}{(z-1)^6}\right]$ $= -z\left[\frac{(z-1)^2\left[(z-1)(2z+1) - 3(z^2+z)\right]}{(z-1)^6}\right]$ $= -z\left[\frac{2z^2 - 2z + z - 1 - 3z^2 - 3z}{(z-1)^4}\right]$ $= -z\left[\frac{-z^2 - 4z - 1}{(z-1)^4}\right]$ $Z\left[n^3\right] = \frac{z(z^2+4z+1)}{(z-1)^4}$ $(1) \Rightarrow \quad Z\left[n(n-1)(n-2)\right] = \frac{z(z^2+4z+1)}{(z-1)^4} - 3\frac{z+z^2}{(z-1)^3} + 2\frac{z}{(z-1)^2}$
4.	**If** $U(z) = \frac{2z^2+5z+14}{(z-1)^4}$ **, evaluate** u_2 and u_3. **Solution:** **Given** $U(z) = F(z) = \frac{2z^2+5z+14}{(z-1)^4}$ We know that $u_0 = f(0) = \lim\limits_{z\to\infty} F(z) = \lim\limits_{z\to\infty}\frac{2z^2+5z+14}{(z-1)^4} = \lim\limits_{z\to\infty}\frac{z^2\left(2+\frac{5}{z}+\frac{14}{z^2}\right)}{z^4\left(1-\frac{1}{z}\right)^2}$ $u_0 = f(0) = 0 \quad \because \frac{1}{\infty} = 0$

	$u_1 = f(1) = \lim_{z\to\infty}\left[zF(z) - zf(0)\right]$ $= \lim_{z\to\infty}\left[\frac{z(2z^2+5z+14)}{(z-1)^4} - z(0)\right]$ $= \lim_{z\to\infty}\left[\frac{z^3\left(2+\frac{5}{z}+\frac{14}{z^2}\right)}{z^4\left(1-\frac{1}{z}\right)^4} - 0\right]$ $u_1 = f(1) = 0 \quad \because \frac{1}{\infty} = 0$ $u_2 = f(2) = \lim_{z\to\infty}\left[z^2F(z) - z^2 f(0) - zf(1)\right]$ $= \lim_{z\to\infty}\left[\frac{z^2(2z^2+5z+14)}{(z-1)^4} - z^2(0) - z(0)\right]$ $= \lim_{z\to\infty}\left[\frac{z^4\left(2+\frac{5}{z}+\frac{14}{z^2}\right)}{z^4\left(1-\frac{1}{z}\right)^4}\right] = \frac{2+0+0}{(1-0)^4} = 2$ $u_2 = f(2) = 2$ $u_3 = f(3) = \lim_{z\to\infty}\left[z^3F(z) - z^3 f(0) - z^2 f(1) - zf(2)\right]$ $= \lim_{z\to\infty}\left[\frac{z^3(2z^2+5z+14)}{(z-1)^4} - z^3(0) - z^2(0) - z(2)\right]$ $= \lim_{z\to\infty}\left[\frac{z^3(2z^2+5z+14)}{(z-1)^4} - 2z\right]$ $= \lim_{z\to\infty} z^3\left(\frac{(2z^2+5z+14)}{(z-1)^4} - \frac{2}{z^2}\right)$ $= \lim_{z\to\infty} z^3\left(\frac{z^2(2z^2+5z+14) - 2(z-1)^4}{z^2(z-1)^4}\right) \because (a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^3$ $= \lim_{z\to\infty} z^3\left(\frac{(2z^4+5z^3+14z^2) - 2(z^4-4z^3+6z^2-4z+1)}{z^2(z-1)^4}\right)$ $= \lim_{z\to\infty} z^3\left(\frac{2z^4+5z^3+14z^2-2z^4+8z^3-12z^2+8z-2}{z^2(z-1)^4}\right)$ $= \lim_{z\to\infty} z^3\left(\frac{13z^3+2z^2+8z-2}{z^2(z-1)^4}\right)$ $= \lim_{z\to\infty}\frac{z^6\left(13+\frac{2}{z}+\frac{8}{z^2}-\frac{2}{z^3}\right)}{z^6\left(1-\frac{1}{z}\right)^4} = \lim_{z\to\infty}\frac{\left(13+\frac{2}{z}+\frac{8}{z^2}-\frac{2}{z^3}\right)}{\left(1-\frac{1}{z}\right)^4} = \frac{13+0+0-0}{(1-0)^4}$ $u_3 = f(3) = 13$
5.	**State and prove initial and final value theorem of Z-transform.** **Initial value theorem:** **If** $Z\left[f(n)\right] = F(z)$ **then** $f(0) = \lim_{z\to\infty} F(z)$ **Proof:**

	We know that

We know that

$$Z\left[f(n)\right]=\sum_{n=0}^{\infty} f(n)z^{-n}$$

$$\lim_{z\to\infty} F(z)=\lim_{z\to\infty}\sum_{n=0}^{\infty} f(n)\left(\frac{1}{z}\right)^{n}$$

$$=\lim_{z\to\infty}\left[f(0)\left(\frac{1}{z}\right)^{0}+f(1)\left(\frac{1}{z}\right)^{1}+f(2)\left(\frac{1}{z}\right)^{2}+\ldots\right]$$

$$\lim_{z\to\infty} F(z)=f(0)\qquad \because \frac{1}{\infty}=0$$

Final value theorem:

If $Z\left[f(n)\right]=F(z)$ **then** $\lim_{n\to\infty} f(n)=\lim_{z\to 1}(z-1)F(z)$

Proof:

$$Z\left[f(n)\right]=\sum_{n=0}^{\infty} f(n)z^{-n}\quad -----(1)$$

$$Z\left[f(n+1)\right]=\sum_{n=0}^{\infty} f(n+1)z^{-n}\quad -----(2)$$

$(1)-(2)\Rightarrow$

$$Z\left[f(n+1)\right]-Z\left[f(n)\right]=\sum_{n=0}^{\infty} f(n+1)z^{-n}-\sum_{n=0}^{\infty} f(n)z^{-n}$$

$$\left[zF(z)-zf(0)\right]-F(z)=\sum_{n=0}^{\infty}\left[f(n+1)-f(n)\right]z^{-n}$$

$$\lim_{z\to 1}\left[(z-1)F(z)-zf(0)\right]=\lim_{z\to 1}\sum_{n=0}^{\infty}\left[f(n+1)-f(n)\right]z^{-n}$$

$$\lim_{z\to 1}\left[(z-1)F(z)\right]-f(0)=\sum_{n=0}^{\infty}\left[f(n+1)-f(n)\right]$$

$$\lim_{z\to 1}\left[(z-1)F(z)\right]-f(0)=\left[\cancel{f(1)}-f(0)\right]+\left[\cancel{f(2)}-\cancel{f(1)}\right]+\ldots+\left[f(n+1)-\cancel{f(n)}\right]+\ldots\infty$$

$$\lim_{z\to 1}\left[(z-1)F(z)\right]\cancel{-f(0)}=\cancel{-f(0)}+f(n+1)+\ldots\infty$$

$$\lim_{z\to 1}\left[(z-1)F(z)\right]=\lim_{n\to\infty} f(n)\qquad \because f(n+1)=f(n)\ \text{when } n\to\infty$$

Hence proved