

1.5 STEADY STATE ANALYSIS OF RLC CIRCUITS

An understanding of the natural response of the series RLC circuit is a necessary background for future studies in filter design and communications networks.

Consider the series RLC circuit shown in Fig. 3.4.1. The circuit is being excited by the energy initially stored in the capacitor and inductor. The energy is represented by the initial capacitor voltage V_0 and initial inductor current I_0 . Thus, at $t = 0$,

$$v(0) = \frac{1}{C} \int_{-\infty}^0 i dt = V_0$$

$$i(0) = I_0$$

Applying KVL around the loop in Fig. 8.8,

$$Ri + L \frac{di}{dt} + \frac{1}{C} \int_{-\infty}^t i dt = 0$$

To eliminate the integral, we differentiate with respect to t and rearrange terms. We get

$$\frac{d^2i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{i}{LC} = 0$$

This is a *second-order differential equation* and is the reason for calling the RLC circuits in this chapter second-order circuits. Our goal is to solve. To solve such a second-order differential equation requires that we have two initial conditions, such as the initial value of i and its first derivative or initial values of some i and v . The initial value of i is given. We get the initial value of the derivative of i from Eqs.

$$Ri(0) + L \frac{di(0)}{dt} + V_0 = 0$$

or

$$\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0)$$

With the two initial conditions in Eqs. (8.2b) and (8.5), we can now Our experience in the preceding chapter on first-order circuits suggests that the solution is of exponential form. So we let

$$i = Ae^{st}$$

where A and s are constants to be determined. and carrying out the necessary differentiations, we obtain

$$As^2e^{st} + \frac{AR}{L}se^{st} + \frac{A}{LC}e^{st} = 0$$

or

$$Ae^{st} \left(s^2 + \frac{R}{L}s + \frac{1}{LC} \right) = 0$$

Since $i = Ae^{st}$ is the assumed solution we are trying to find, only the expression in parentheses can be zero:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

This quadratic equation is known as the *characteristic equation* of the differential since the roots of the equation dictate the character

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

A more compact way of expressing the roots is

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

where

$$\alpha = \frac{R}{2L}, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

Problem 1:

In Fig. 3.4.1, $R = 40 \ \Omega$, $L = 4 \text{ H}$, and $C = 1/4 \text{ F}$. Calculate the characteristic roots of the circuit. Is the natural response overdamped, underdamped, or critically damped?

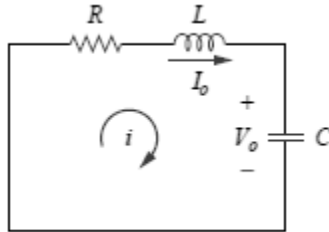


Fig. 1.5.1 For problem 1.

[Source: "Fundamentals of Electric Circuits" by Charles K. Alexander, page: 301]

Solution:

We first calculate

$$\alpha = \frac{R}{2L} = \frac{40}{2(4)} = 5, \quad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4 \times \frac{1}{4}}} = 1$$

The roots are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -5 \pm \sqrt{25 - 1}$$

or

$$s_1 = -0.101, \quad s_2 = -9.899$$