Method of Undetermined Coefficient

The given differential equation is $F(D)y = f(x)$

To find the particular integral (P.I) of the given equation, we have to assume a trial solution that contains unknown constants. This unknown constants are to be determined by substitution in the given equation and the trial solution depends on the given function $f(x)$.

Sl.No	Function $f(x)$	Choice of P.I
1	ke^{px}	Ce^{px}
2	$ksin(ax+b)$ (or) $kcos(ax+b)$	$C_1sin(ax+b) + C_2cos(ax+b)$
3	$ke^{px}sin(ax+b)$ (or) $ke^{px}cos(ax+b)$	$C_1e^{px}sin(ax+b) + C_2e^{px}cos(ax+b)$
4	$kx^m where\ m = 0,1,2, \ldots \ldots$	$C_0 + C_1x + C_2x^2 + \cdots \ldots \ldots C_mx^m$

Straight Case:

If the R.H.S function $f(x)$ is not a member of the solution set, then choose, P.I,(y_p) from the above table depending on the nature of $f(x)$

Sum Case:

When the R.H.S $f(x)$ is a Combination (sum) of the functions in column "2" of the table, then P.I is chosen as a Combination of the corresponding function in third column and proceed as in straight case.

Modified Case:

When any term of $f(x)$ is a member of the solution set S, then the method fails. If we choose y_p from the table. In such cases the choice from the table should be modified as follows.

a) If a term u of $f(x)$ is also a term of the C.F then the choice from the table corresponding to u should be multiplied by

* x if u corresponds to a simple root of C.F

* x^2 if u corresponds to a double root of C.F

* x^s if u corresponds to a s-fold root of C.F

b) Suppose $x^r u$ is a term $f(x)$ and u is a term of C.F corresponding to an S-fold root then the choice from the table corresponding to $x^r u$ should be multiplied by x^s.

Problems based on, to find the particular integral by the method of undetermined coefficients

Type I : Straight Case:

Example : 1

Solve $(D^2 + 1)y = e^{3x}$

Solution:

$$y'' + 9y = e^{3x} \quad \dots (1)$$

Auxiliary Equation is $m^2 + 1 = 0$

$$m^2 = \pm i$$

C.F= $e^{0x}[Acosx + Bsinx] \quad \dots (2)$

Here the solution set $S = \{cosx, sinx\}$

R.H.S of (1) is not a member of S

Choose P.I $y_p = ce^{3x} \quad \dots (3)$

$$y_p' = 3Ce^{3x}$$

$$y_p'' = 9Ce^{3x}$$

(1) $\Rightarrow$ $\quad 9Ce^{3x} + Ce^{3x} = e^{3x}$

$$10Ce^{3x} = e^{3x} \Rightarrow C = \frac{1}{10}$$

(3) $\Rightarrow$ $\quad y_{p} = \frac{1}{10}e^{3x}$

The general solution is $y = C.F + P.I$

$$y = Acosx + Bsinx + \frac{1}{10}e^{3x}$$

Example : 2

Solve $(\boldsymbol{D^2 - 3D + 2})\boldsymbol{y} = \boldsymbol{6e^{3x}}$

Solution:

Given $y'' - 3y' + 2y = 6e^{3x} \dots (1)$

Auxiliary Equation is $m^2 - 3m + 2 = 0$

$$m = 1,2$$

$$C.F = Ae^x + Be^{2x} \dots (2)$$

Here the Solution Set $S = \{e^x, e^{2x}\}$

R.H.S of (1) is not a member of S

$$P.I \quad y_p = Ce^{3x} \dots (3)$$

$$y_p{}' = 3Ce^{3x}$$

$$y_p{}' = 9Ce^{3x}$$

(1) $\Rightarrow$ $9Ce^{3x} - 9Ce^{3x} + 2Ce^{3x} = 6e^{3x}$

$$2Ce^{3x} = 6e^{3x}$$

$$2C = 6$$

$$C = 3$$

$$(3) \Rightarrow y_p = 3e^{3x}$$

The general solution is $y = y_{c+}y_p$

$$y = Ae^x + Be^{2x} + 3e^{3x}$$

Example :3

Solve $(D^2 - 2D + 1)y = cosx$

Solution:

Given $y" - 3y' - 2y' = cosx \dots (1)$

Auxiliary Equation is $m^2 - 2m + 1 = 0$

$$(m-1)^2 = 0$$

$$m = 1,1$$

$$C.F = (A + Bx)e^x \ldots (2)$$

The Solution Set $S = \{e^x, xe^x\}$

R.H.S of (1) is not a member of S

$$P.I \quad y_p = C_1 sinx + C_2 cosx \ldots (3)$$

$$y_p' = C_1 cos\, x - C_2 sinx$$

$$y_p" = -C_1 sinx - C_2 cosx$$

(1) $\Rightarrow$ $-C_1 sinx - C_2 cosx - 2C_1 cosx + 2C_2 Sinx + C_1 Sinx + C_2 cosx = cosx$

$$-2C_1 cosx + 2C_2 Sinx = cosx$$

Equating the coefficients of

$cosx$: $-2C_1 = 1 \Rightarrow C_1 = \frac{-1}{2}$

$Sinx$: $2C_2 = 0 \Rightarrow C_2 = 0$

(3) $\Rightarrow$ P.I $y_p = \frac{-1}{2} sinx$

The general solution is $y = C.F + P.I$

$$y = (Ax + Bx)e^x - \frac{1}{2} sinx$$

Example :4

Solve $y'' + 2y' + 5y = 6sin3x + 7cos3x$

Solution:

Given $y'' + 2y' + 5y = 6Sin3x + 7cos3x \dots (1)$

Auxiliary Equation is $m^2 + 2m + 5 = 0$

$$m = -1 \pm 2i$$

$$C.F = e^{-x}[Acos2x + Bsin2x] \dots (2)$$

Here the Solution Set $S = \{e^{-x}cos2x, e^{-x}sin2x\}$

R.H.S of (1) is not a member of S

$$P.I \quad y_p = C_1 sin3x + C_2 cos3x \dots (3)$$

$$y_p{}' = 3C_1 cos3x - 3C_2 sin3x$$

$$y_p{}'' = -9C_1 sin3x - 9C_2 cos3x$$

$$(1) \Rightarrow \quad -9C_1 sin3x - 9C_2 cos3x + 6C_1 cos3x - 6C_2 Sin3x + 5C_1 Sin3x$$

$$+5C_2 cos3x = 6sin3x + 7cos3x$$

Equating the coefficients of

$sin3x$: $-9C_1 - 6C_2 + 5C_1 = 6 \quad \Rightarrow \quad -4C_1 - 6C_2 = 6 \dots (4)$

$cos3x$: $-9C_2 + 6C_1 + 5C_2 = 7 \quad \Rightarrow \quad 6C_1 - 4C_2 = 7 \dots (5)$

Solving (4) & (5) $\quad C_1 = \frac{9}{26} \quad \& \quad C_2 = \frac{-16}{13}$

$$(3) \Rightarrow \quad y_p = \frac{9}{26} sin3x - \frac{16}{13} cos3x$$

The general solution is $y = C.F + P.I$

$$y = e^{-x}[Acos2x + Bsin2x] + \frac{9}{26}sin3x - \frac{16}{13}cos3x$$

Example :5

Solve $(D^2 + 2D + 5)y = x^2$

Solution:

Given $y'' + 2y' + 5y = x^2$ ... (1)

Auxiliary Equation is $m^2 + 2m + 5 = 0$

$$m = \frac{-2 \pm \sqrt{4-20}}{20}$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm 4i}{2} \quad = -1 \pm 2i$$

$C.F = e^{-x}[Acos2x + Bsin2x]$ ... (2)

Here the Solution Set $S = \{e^{-x}cos2x, e^{-x}sin2x\}$

R.H.S of (1) is not a member of S

Choose $P.I \quad y_p = C_0 + C_1x + C_2x^2$ ... (3)

$$y_p' = C_1 + 2C_2x$$

$$y_p'' = 2C_2$$

(1) $\Rightarrow \quad 2C_2 + 2C_1 + 4C_2x + 5C_0 + 5C_1x + 5C_2x^2 = x^2$

Equating the coefficients of

x^2: $\quad 5C_2 = 1 \quad \Rightarrow \quad C_2 = \frac{1}{5}$

x: $\quad 4C_2 + 5C_1 = 0$

$$4\left(\frac{1}{5}\right) + 5C_1 = 0$$

$$5C_1 = \frac{-4}{5}$$

$$C_1 = \frac{-4}{25}$$

Constant:

$$2C_2 + 2C_1 + 5C_0 = 0$$

$$2\left(\frac{1}{5}\right) - 2\left(\frac{4}{25}\right) + 5C_0 = 0$$

$$5C_0 = \frac{-2}{5} + \frac{8}{25}$$

$$5C_0 = \frac{-2}{25}$$

$$C_0 = \frac{-2}{125}$$

(3) $\Rightarrow \quad y_p = \frac{-2}{125} + \left(\frac{-4}{25}\right)x + \frac{1}{5}x^2$

The general solution is $y = C.F + P.I$

$$y = e^{-x}[A\cos 2x + B\sin 2x] - \frac{2}{125} - \frac{4}{25}x + \frac{x^2}{2}$$

Type II : Sum Case:

Example :6

Solve $(D^2 + 2D + 5)y = 2x^2 + 3e^{-x}$

Solution:

Given $y'' + 2y' + 4y = 2x^2 + 3e^{-x}$ ... (1)

Auxiliary Equation is $m^2 + 2m + 4 = 0$

$$m = \frac{-2 \pm \sqrt{4-16}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= -1 \pm \sqrt{3}i$$

$$C.F = e^{-x}\left[Acos\sqrt{3}x + Bsin\sqrt{3}x\right] \ldots (2)$$

Here the Solution Set $S = \left\{e^{-x}cos\sqrt{3}x, e^{-x}sin\sqrt{3}x\right\}$

R.H.S of (1) is not a member of S

Choose $P.I \quad y_p = C_0 + C_1x + C_2x^2 + C_3e^{-x} \ldots (3)$

$$y_p' = C_1 + 2C_2x - C_3e^{-x}$$

$$y_p'' = 2C_2 + C_3e^{-x}$$

$$(1) \Rightarrow 2C_2 + C_3e^{-x} + 2C_1 + 4C_2x - 2C_3e^{-x} + 4C_0 + 4C_1x + 4C_2x^2 + 4C_3e^{-x}$$

$$= 2x^2 + 3e^{-x}$$

Equating the coefficients of

x^2 : $\quad 4C_2 = 2$

$$C_2 = \frac{2}{4} = \frac{1}{2}$$

e^{-x} : $\quad 4C_3 + C_3 - 2C_3 = 3$

$$3C_3 = 3$$

$$C_3 = 1$$

x : $\quad 4C_2 + 4C_1 = 0$

$$4\left(\frac{1}{2}\right) + 4C_1 = 0$$

$$4C_1 = \frac{-4}{2}$$

$$C_1 = \frac{-4}{8} \qquad = \frac{-1}{2}$$

Constant: $2C_2 + 2C_1 + 4C_0 = 0$

$$2\left(\frac{1}{2}\right) + 2\left(\frac{-1}{2}\right) + 4C_0 = 0$$

$$C_0 = 0$$

(3) $\Rightarrow$ $y_p = \frac{-1}{2}x + \frac{1}{2}x^2 + e^{-x}$

$$= \frac{-x}{2} + \frac{x^2}{2} + e^{-x}$$

The general solution is $y = C.F + P.I$

$$y = e^{-x}\left[Acos\sqrt{3}x + Bsin\sqrt{3}x\right] - \frac{x}{2} + \frac{x^2}{2} + e^{-x}$$

Example :7

Solve $(D^2 + D - 2)y = x + sinx$

Solution:

Given $y'' + y' - 2y = x + sinx$ ... (1)

Auxiliary Equation is $m^2 + m - 2 = 0$

$$m = 1, -2$$

$C.F = Ae^{x} + Be^{-2x}$ ... (2)

Here the Solution Set $S = \{e^{x}, e^{-2x}\}$

R.H.S of (1) is not a member of S

choose $P.I$ $\quad y_p = C_0 + C_1x + C_2sinx + C_3cosx \ldots (3)$

$$y_p' = C_1 + C_2cosx - C_3sinx$$

$$y_p'' = -C_2sinx - C_3cosx$$

$(1) \Rightarrow \quad -C_2sinx - C_3cosx + C_1 + C_2cosx - C_3Sinx$

$$-2C_0 - 2C_1x - 2C_3Sinx - 2C_3cosx = x + sinx$$

Equating the coefficients of

x: $\quad -2C_1 = 1$

$$C_1 = \frac{-1}{2}$$

$Constant$: $\quad C_1 - 2C_0 = 0$

$$C_0 = \frac{-1}{4}$$

$sinx$: $\quad -C_2 - C_3 - 2C_2 = 1$

$$-3C_2 - C_3 = 1 \ldots (4)$$

$cosx$: $\quad -C_3 + C_2 - 2C_3 = 0$

$$C_2 - 3C_3 = 0 \ldots (5)$$

Solving (4) & (5) we get $\quad C_2 = \frac{-3}{10} \qquad C_3 = \frac{-1}{10}$

$(3) \Rightarrow \quad y_p = \frac{-1}{4} - \frac{1}{2}x - \frac{3}{10}sinx - \frac{1}{10}cosx$

The general solution is $y = C.F + P.I$

$$y = Ae^{x} + Be^{-2x} - \frac{1}{4} - \frac{1}{2}x - \frac{3}{10}sinx \frac{-1}{10}cosx$$

Type: III Modified Case:-

Example:8

Solve $(D^2 + 9)y = cos3x$

Solution:

Given $y" + 9y = cos3x$ ... (1)

Auxiliary Equation is $m^2 + 9 = 0$

$m = \pm 3i$

$C.F = Acos3x + Bsin3x$ ... (2)

Here the solution set $S = \{cos3x, sin3x\}$

R.H.S of (1) is a member of S

Choose $P.I$ $y_p = C_1 sin3x + C_2 cos3x$

Corresponding terms should multiplied by x

$y_p = x[C_1 sin3x + C_2 cos3x]$ ... (3)

$$y_p' = x[3C_1 cos3x - 3C_2 sin3x] + [C_1 cos3x + C_2 sin3x] +$$

$$y_p" = x[-9C_1 sin3x - 9C_2 cos3x] +$$

$$[3C_1 cos3x - 3C_2 sin3x] + 3C_1 cos3x - 3C_2 sin3x$$

(1)⇒ $-9C_1 xsin3x - 9C_2 xcos3x +$

$$6C_1 cos3x - 6C_2 sin3x + 9C_1 xsinx + 9C_2 xcos3x = cos3x$$

Equating the coefficients of

$cos3x$: $6C_1 = 1$

$$C_1 = \frac{1}{6}$$

$sin3x$: $-6C_2 = 0$

$$C_2 = 0$$

(3) ⇒ $y_p = x\left[\frac{1}{6}sin3x\right]$

The general solution is $y = C.F + P.I$

$$y = Acos3x + Bsin3x + \frac{x}{6}sin3x$$

Example:9

Solve $y" + y' - 6y = 10e^{2x} - 18e^{3x} - 6x - 11$

Solution:

$$y" + y' - 6y = 10e^{2x} - 18e^{3x} - 6x - 11 \ldots (1)$$

Auxiliary Equation is $m^2 + m - 6 = 0$

$$m = 2, -3$$

$$C.F = Ae^{2x} + Be^{-3x} \ldots (2)$$

Here the Solution Set $S = \{e^{2x}, e^{-3x}\}$

R.H.S of (1) is not a member of S

Choose $P.I$ $y_p = C_1e^{2x} + C_2e^{3x} + C_3x + C_4$

Corresponding terms should multiplied by x

$$y_p = x(C_1e^{2x}) + C_2e^{3x} + C_3x + C_4 \ldots (3)$$

$$y_p{}' = C_1x2e^{2x} + C_1e^{2x} + 3C_2e^{3x} + C_3$$

$$= (2C_1x + C_1)e^{2x} + 3C_2e^{2x} + C_3$$

$$y_p" = (2C_1x + C_1)2e^{2x} + e^{2x}(2C_1) + 9C_2e^{3x}$$

$$= 2e^{2x}[2C_1x + 2C_1] + 9C_2e^{3x}$$

$$= 4e^{2x}(x + 1)C_1 + 9C_2e^{3x}$$

(1) $\Rightarrow\ 4(x+1)C_1e^{3x} + 9C_2e^{3x} + (2x+1)C_1e^{3x} + 3C_2e^{3x}$

$$-6xC_1e^{3x} - 6xC_2e^{3x} - 6C_3x - 6C_4$$

$$= 10e^{2x} - 18e^{2x} - 6x - 11$$

$$(4x + 4 + 2x + 1 - 6x)C_1e^{2x} + 6C_2e^{3x} + C_3 - 6C_3x - 6C_4$$

$$= 10e^{2x} - 18e^{3x} - 6x - 11$$

Equating the coefficients of

e^{2x} : $5C_1 = 10$

$C_1 = 2$

e^{3x} : $6C_2 = 18$

$C_2 = -3$

x : $-6C_3 = -6$

$C_3 = 1$

Constant : $C_3 - 6C_4 = -11$

$$C_4 = 2$$

(3) $\Rightarrow$ $y_p = 2xe^{2x} - 3e^{3x} + x + 2$

The general solution is $y = C.F + P.I$

$$y = Ae^{2x} + Be^{-3x} + 2xe^{2x} - 3e^{3x} + x + 2$$