4.4 SELF RECIPROCAL UNDER FOURIER TRANSFORM

Self-reciprocal:

If a transformation of a function *f(x)* is equal to *f(s)* then the function *f(x)* is called **self-reciprocal.**

Find the Fourier transform of $e^{-a^2x^2}$ Hence prove that $e^{\frac{-x^2}{2}}$ is self-reciprocal with respect to Fourier Transforms.

Solution:

The Fourier transform f(x) is

$$F\left[f(x)\right]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}dx$$

$$F\left[e^{-a^2x^2}\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-a^2x^2}e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-a^2x^2+isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\left(a^2x^2-isx\right)}dx$$

$$(A-B)^2=A^2-2AB+B^2$$

$$2AB=isx$$

Here $A=ax\ ,B=\frac{is}{2a}$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\left[(ax)^2-isx+\left(\frac{is}{2a}\right)^2-\left(\frac{is}{2a}\right)^2\right]}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\left(ax-\frac{is}{2a}\right)^2}e^{\left(\frac{is}{2a}\right)^2}dx$$

$$=\frac{1}{\sqrt{2\pi}}e^{\left(\frac{is}{2a}\right)^2}\int_{-\infty}^{\infty}e^{-\left(ax-\frac{is}{2a}\right)^2}dx$$

Let $u=ax-\frac{is}{2a}\ \Rightarrow du=adx\ \Rightarrow dx=\frac{du}{a}\ ;\ u:-\infty\, to\, \infty$

$$=\frac{1}{\sqrt{2\pi}}e^{\frac{i^2s^2}{4a^2}}\int_{-\infty}^{\infty}e^{-u^2}\frac{du}{a}$$

$$=\frac{1}{a\sqrt{2\pi}}e^{\frac{-s^2}{4a^2}}\int_{-\infty}^{\infty}e^{-u^2}du\quad \because i^2=-1$$

$$=\frac{1}{a\sqrt{2\pi}}e^{\frac{-s^2}{4a^2}}2\int_{0}^{\infty}e^{-u^2}du\quad \because e^{-u^2}\text{ is an even function}$$

$$=\frac{1}{a\sqrt{2}\sqrt{\pi}}e^{\frac{-s^2}{4a^2}}2\frac{\sqrt{\pi}}{2}\qquad \because \int_{0}^{\infty}e^{-u^2}du=\frac{\sqrt{\pi}}{2}$$

$$\boxed{F\left[e^{-a^2x^2}\right]=\frac{1}{a\sqrt{2}}e^{\frac{-s^2}{4a^2}}}\qquad ------(1)$$

Deduction:

To prove $e^{\frac{-x^2}{2}}$ is self-reciprocal

It is enough to prove that $F\left[e^{-\frac{x^2}{2}}\right]$ is $e^{-\frac{s^2}{2}}$

Put $a=\frac{1}{\sqrt{2}}$ in (1)

$$F\left[e^{-\left(\frac{1}{\sqrt{2}}\right)^2 x^2}\right]=\frac{1}{\left(\frac{1}{\sqrt{2}}\right)\sqrt{2}}e^{\frac{-s^2}{4\left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$F\left[e^{-\frac{x^2}{2}}\right]=e^{\frac{\frac{-s^2}{4}}{2}}$$

$$\boxed{F\left[e^{-\frac{x^2}{2}}\right]=e^{\frac{-s^2}{2}}}$$

$\therefore e^{\frac{-x^2}{2}}$ is self reciprocal.

Find the Fourier transform of $e^{\frac{-x^2}{2}}$.

(or) Show that $e^{\frac{-x^2}{2}}$ is self-reciprocal with respect to Fourier Transforms.

Solution:

Let $f(x)=e^{-\frac{x^2}{2}}$

Assume $f(x)=e^{-a^2x^2}$ where $a=\frac{1}{\sqrt{2}}$

The Fourier transform f(x) is

$$F\left[f(x)\right]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F\left[e^{-a^2x^2}\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2}e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2+isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(a^2x^2-isx\right)}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left[(ax)^2-isx+\left(\frac{is}{2a}\right)^2-\left(\frac{is}{2a}\right)^2\right]}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2}e^{\left(\frac{is}{2a}\right)^2}dx$$

$$=\frac{1}{\sqrt{2\pi}}e^{\left(\frac{is}{2a}\right)^2}\int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2}dx$$

$(A-B)^2=A^2-2AB+B^2$

$2AB=isx$

Here $A=ax\ ,B=\frac{is}{2a}$

Let $u=ax-\frac{is}{2a}\ \Rightarrow du=adx\ \Rightarrow dx=\frac{du}{a}\ ;\ u:-\infty\,to\,\infty$

$$=\frac{1}{\sqrt{2\pi}}e^{\frac{i^2s^2}{4a^2}}\int_{-\infty}^{\infty} e^{-u^2}\frac{du}{a}$$

$$= \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} du \quad \because i^2 = -1$$

$$= \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}} 2\int_{0}^{\infty} e^{-u^2} du \quad \because e^{-u^2} \text{ is an even function}$$

$$= \frac{1}{a\sqrt{2}\sqrt{\pi}} e^{\frac{-s^2}{4a^2}} 2\frac{\sqrt{\pi}}{2} \qquad \because \int_{0}^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}$$

$$\boxed{F\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}} e^{\frac{-s^2}{4a^2}}} \qquad ------(1)$$

Deduction:

To prove $e^{\frac{-x^2}{2}}$ is self-reciprocal

It is enough to prove that $F\left[e^{-\frac{x^2}{2}}\right]$ is $e^{-\frac{s^2}{2}}$

Put $a = \frac{1}{\sqrt{2}}$ in (1)

$$F\left[e^{-\left(\frac{1}{\sqrt{2}}\right)^2 x^2}\right] = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)\sqrt{2}} e^{\frac{-s^2}{4\left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$F\left[e^{-\frac{x^2}{2}}\right] = e^{\frac{\frac{-s^2}{4}}{\frac{1}{2}}}$$

$$\boxed{F\left[e^{-\frac{x^2}{2}}\right] = e^{\frac{-s^2}{2}}}$$

$\therefore e^{\frac{-x^2}{2}}$ is self-reciprocal.

Find the Fourier cosine transform of $e^{-a^2x^2}$ Hence find $F_s\left[xe^{-a^2x^2}\right]$.

Solution:

Let $f(x) = e^{-a^2x^2}$

The Fourier cosine transform f(x) is

$$F_c\left[f(x)\right] = F_c(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x)\cos sx\, dx \qquad \because \int_{0}^{\infty} f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-a^2x^2} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_{-\infty}^{\infty} e^{-a^2x^2} \cos sx\, dx$$

$$F_c\left[f(x)\right] = \text{R.P. of } \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx \qquad \because \cos sx = \text{R.P. of } e^{isx}$$

$$F_c\left[f(x)\right] = \text{R.P. of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx$$

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$$= \text{R.P.of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2+isx} dx$$

$$= \text{R.P.of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(a^2x^2-isx\right)} dx$$

$$= \text{R.P.of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[(ax)^2-isx+\left(\frac{is}{2a}\right)^2-\left(\frac{is}{2a}\right)^2\right]} dx$$

$$= \text{R.P.of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2} e^{\left(\frac{is}{2a}\right)^2} dx$$

$$= \text{R.P.of } \frac{1}{\sqrt{2\pi}} e^{\left(\frac{is}{2a}\right)^2} \int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2} dx$$

$(a-b)^2 = a^2 - 2ab + b^2$

$-2ab = isx$

Here $a = ax$

$-2axb = -isx \Rightarrow b = \frac{is}{2a}$

Let $u = ax - \frac{is}{2a} \Rightarrow du = adx \Rightarrow dx = \frac{du}{a}$; $u : -\infty \, to \, \infty$

$$= \text{R.P.of } \frac{1}{\sqrt{2\pi}} e^{\frac{i^2s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{a}$$

$$= \text{R.P.of } \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} du \quad \because i^2 = -1$$

$$= \text{R.P.of } \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}} 2\int_{0}^{\infty} e^{-u^2} du \quad \because e^{-u^2} \text{ is an even function}$$

$$= \text{R.P.of } \frac{1}{a\sqrt{2}\sqrt{\pi}} e^{\frac{-s^2}{4a^2}} 2\frac{\sqrt{\pi}}{2} \qquad \because \int_{0}^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}$$

$$F\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}} e^{\frac{-s^2}{4a^2}} \qquad ------(1)$$

Deduction:

$$F_s\left[xf(x)\right] = -\frac{d}{ds}\left\{F_c\left[f(x)\right]\right\} = -\frac{d}{ds}\left[F_c(s)\right]$$

$$F_s\left[xe^{-a^2x^2}\right] = -\frac{d}{ds}\left\{F_c\left[e^{-a^2x^2}\right]\right\}$$

$$= -\frac{d}{ds}\left[\frac{1}{a\sqrt{2}} e^{\frac{-s^2}{4a^2}}\right]$$

$$= -\frac{1}{a\sqrt{2}} e^{\frac{-s^2}{4a^2}} \left(\frac{-2s}{4a^2}\right)$$

$$F_s\left[xe^{-a^2x^2}\right] = \frac{s}{2\sqrt{2}a^3} e^{\frac{-s^2}{4a^2}}$$

Solve for *f(x)*, the integral equation $\int_{0}^{\infty} f(x)\sin sx dx = \begin{cases} 1, & 0 \le s < 1 \\ 2, & 1 \le s < 2. \\ 0, & s \ge 2 \end{cases}$

Solution:

Given $\int_0^\infty f(x)\sin sx dx = \begin{cases} 1, & 0 \le s < 1 \\ 2, & 1 \le s < 2. \\ 0, & s \ge 2 \end{cases}$ ------(1)

We know that

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\, dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} F_s^{-1}\left[\begin{cases} 1, & 0 \le s < 1 \\ 2, & 1 \le s < 2 \\ 0, & s \ge 2 \end{cases}\right] \qquad F^{-1}[F_s(s)] = f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\, ds$$

$$f(x) = \sqrt{\frac{2}{\pi}}\sqrt{\frac{2}{\pi}}\left[\int_0^1 1\sin sx\, ds + \int_1^2 2\sin sx\, ds + \int_2^\infty 0\sin sx\, ds\right]$$

$$= \frac{2}{\pi}\left[\int_0^1 1\sin sx\, ds + \int_1^2 2\sin sx\, ds\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{-\cos sx}{x}\right)_0^1 + 2\left(\frac{-\cos sx}{x}\right)_1^2\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{-\cos x}{x} + \frac{\cos 0}{x}\right) + 2\left(\frac{-\cos 2x}{x} + \frac{\cos x}{x}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{-\cos x}{x} + \frac{1}{x} - 2\frac{\cos 2x}{x} + 2\frac{\cos x}{x}\right]$$

$$= \frac{2}{\pi x}[1 - \cos x - 2\cos 2x + 2\cos x]$$

$$f(x) = \frac{2}{\pi x}[1 + \cos x - 2\cos 2x]$$

Find the Fourier cosine and sine transform of x^{n-1}. Hence show that $\frac{1}{\sqrt{x}}$ is self-reciprocal under Fourier cosine and sine transforms.

Solution:

By definition of Gamma integral

$$\int_0^\infty e^{-ax}x^{n-1}\, dx = \frac{\Gamma n}{a^n}, \quad a > 0, n > 0$$

Put $a = is$

$$\int_0^\infty e^{-isx}x^{n-1}\, dx = \frac{\Gamma n}{(is)^n}, \quad a > 0, n > 0$$

$$\int_0^\infty x^{n-1}e^{-isx}\, dx = \frac{\Gamma n}{i^n s^n}$$

$$= \frac{\Gamma n}{s^n}(-i)^n$$

$$= \frac{\Gamma n}{s^n}\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)^n \qquad \because e^{-i\frac{\pi}{2}} = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i$$

$$= \frac{\Gamma n}{s^n}\left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right) \qquad \because \text{by Demorive's theorem} \left(\cos\theta \pm i\sin\theta\right)^n = \cos n\theta \pm i\sin n\theta$$

$$\int_0^\infty x^{n-1}(\cos sx - i\sin sx)\, dx = \frac{\Gamma n}{s^n}\left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right)$$

$$\int_0^\infty x^{n-1}\cos sx\,dx - i\int_0^\infty x^{n-1}\sin sx\,dx = \frac{\Gamma n}{s^n}\cos\frac{n\pi}{2} - i\frac{\Gamma n}{s^n}\sin\frac{n\pi}{2}$$

Equating real and imaginary parts on both sides

$$\int_0^\infty x^{n-1}\cos sx\,dx = \frac{\Gamma n}{s^n}\cos\frac{n\pi}{2} \qquad \int_0^\infty x^{n-1}\sin sx\,dx = \frac{\Gamma n}{s^n}\sin\frac{n\pi}{2}$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\cos sx\,dx = \sqrt{\frac{2}{\pi}}\frac{\Gamma n}{s^n}\cos\frac{n\pi}{2} \qquad \sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\sin sx\,dx = \sqrt{\frac{2}{\pi}}\frac{\Gamma n}{s^n}\sin\frac{n\pi}{2}$$

$$\boxed{F_c\left[x^{n-1}\right] = \sqrt{\frac{2}{\pi}}\frac{\Gamma n}{s^n}\cos\frac{n\pi}{2}} \qquad \boxed{F_s\left[x^{n-1}\right] = \sqrt{\frac{2}{\pi}}\frac{\Gamma n}{s^n}\sin\frac{n\pi}{2}}$$

Deduction:

To prove $\frac{1}{\sqrt{x}}$ is self-reciprocal under Fourier cosine and sine transforms.

It is enough to prove that $F_c\left[\frac{1}{\sqrt{x}}\right] = \frac{1}{\sqrt{s}}$ and $F_s\left[\frac{1}{\sqrt{x}}\right] = \frac{1}{\sqrt{s}}$

We know that

$$F_c\left[x^{n-1}\right] = \sqrt{\frac{2}{\pi}}\frac{\Gamma n}{s^n}\cos\frac{n\pi}{2} \qquad F_s\left[x^{n-1}\right] = \sqrt{\frac{2}{\pi}}\frac{\Gamma n}{s^n}\sin\frac{n\pi}{2}$$

Put $n = \frac{1}{2}$

$$F_c\left[x^{\frac{1}{2}-1}\right] = \sqrt{\frac{2}{\pi}}\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}\cos\frac{\pi}{4} \qquad F_s\left[x^{\left(\frac{1}{2}-1\right)}\right] = \sqrt{\frac{2}{\pi}}\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}\sin\frac{\pi}{4}$$

$$F_c\left[x^{-\frac{1}{2}}\right] = \frac{\sqrt{2}}{\sqrt{\pi}}\frac{\sqrt{\pi}}{\sqrt{s}}\frac{1}{\sqrt{2}} \qquad F_s\left[x^{-\frac{1}{2}}\right] = \frac{\sqrt{2}}{\sqrt{\pi}}\frac{\sqrt{\pi}}{\sqrt{s}}\frac{1}{\sqrt{2}} \qquad \because \cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \text{ and } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\boxed{F_c\left[\frac{1}{\sqrt{x}}\right] = \frac{1}{\sqrt{s}}} \qquad \boxed{F_s\left[\frac{1}{\sqrt{x}}\right] = \frac{1}{\sqrt{s}}}$$

$\therefore$ $\frac{1}{\sqrt{x}}$ is self-reciprocal under Fourier cosine and sine transforms.

Find the function $f(x)$ if its sine transform is $\frac{e^{-as}}{s}$

Solution:

Given $F_s\left[f(x)\right] = F_s(s) = \frac{e^{-as}}{s}$

$$f(x) = F^{-1}\left[F_s(s)\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\,ds$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\sin sx\,ds$$

Taking diff on both sides w.r.to x

$$\frac{d}{dx}\left[f(x)\right] = \frac{d}{dx}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\sin sx\,ds\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\frac{\partial}{\partial x}\left(\sin sx\right)ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-as}}{\not{s}} \cos sx \times \not{s} \, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-as} \cos sx \; ds$$

$$\frac{d}{dx}\left[f(x) \right] = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + x^2} \right] \quad \because \int_0^\infty e^{-ax} \cos bx \; dx = \frac{a}{a^2 + s^2} \text{ here } a = a, \; b = x$$

Integrating on w.r.to x

$$f(x) = \sqrt{\frac{2}{\pi}} a \int \frac{1}{a^2 + x^2} \quad \because \int_0^\infty \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a} \right)$$

$$= \sqrt{\frac{2}{\pi}} a \frac{1}{a} \tan^{-1}\left(\frac{x}{a} \right)$$

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{x}{a} \right)}$$