4.5 Schrodinger wave equation

4.5.1 Schrodinger Time Independent wave equation

Consider a wave associated with a moving particle. Let *x*, *y*, *z* be the coordinate of the particle and Ψ is a wave function for de – Broglie at any instant of time t.

The classical differential equation for wave motion is given by

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2} \text{-------------------(1)}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}$$

Laplacian operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

(1) gives $\nabla^2\Psi = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}$ ------------(2)

The solution of equation (2) becomes

$$\Psi(x, y, z, t) = \Psi_0(x, y, z)e^{-i\omega t} \text{------------(3)}$$

Differentiating (3) twice w.r.t time't'

$$\frac{\partial\Psi}{\partial t} = -i\,\Psi_0 e^{-i\omega t}$$

$$\frac{\partial^2\Psi}{\partial t^2} = (-i\,w)(-iw)\Psi_0 e^{-i\omega t} \text{---------------(4)}$$

$$\frac{\partial^2\Psi}{\partial t^2} = -\,w^2\,\Psi \text{------------------------------(5)}$$

Substitute (5) in (2)

$\nabla^2 = -\frac{1}{v^2}(w^2\, \Psi)$

$\nabla^2 = -\frac{w^2}{v^2}\Psi$--------------(6)

w.k.t

$\omega = 2\,\pi\upsilon$; but $v = \upsilon\,\lambda$

$\omega = 2\,\pi\frac{\upsilon}{\lambda}$ $\upsilon = \frac{\upsilon}{\lambda}$

$\frac{\omega}{v} = \frac{2\,\pi}{\lambda}$

$\frac{w^2}{v^2} = \frac{4\pi^2}{\lambda^2}$--------------(7)

Sub (7) in (6)

$\nabla^2\,\Psi = -\frac{4\pi^2}{\lambda^2}\Psi$

$\nabla^2\,\Psi + \frac{4\pi^2}{\lambda^2}\Psi = 0$--------(8)

According to De-Broglie's theory $\lambda = \frac{h}{mv}$----------(9)

Where m - mass of particle

v - velocity

sub (9) in (8)

$$\nabla^2 \Psi + \frac{4\pi^2}{(\frac{h}{mv})^2}\Psi = 0$$

$$\nabla^2 \Psi + \frac{4\pi^2 v^2 m^2}{h^2} = 0 \text{--------}(10)$$

Taking $\hbar = \frac{h}{2\pi}$; $\frac{1}{\hbar} = \frac{2\pi}{h}$

$$\frac{1}{\hbar^2} = \frac{4\pi^2}{h^2} \text{-----------------}(11)$$

Sub (11) in (10)

$$\nabla^2 \Psi + \frac{v^2 m^2}{h^2} = 0 \text{--------}(12)$$

Total Energy E = V + $\frac{1}{2}mv^2$

2(E-V) = mv^2

Multiply 'm' on both sides

2m(E-V) = m^2v^2-------------(13)

Sub (13) in (12)

$$\nabla^2 \Psi + \frac{2m(E-V)}{\hbar^2}\Psi = 0$$

This is the final expression of Schrodinger time independent wave equation.

4.5.2 Schrodinger Time dependent wave equation:

The differential equation for wave motion is given by

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2}$$

$$\nabla^2 \Psi = \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2} \text{------------}(1)$$

The solution of equation (1) becomes

$$\Psi(x, y, z, t) = \Psi_0(x, y, z)e^{-i\omega t} \text{---------}(2)$$

Differentiating (2) twice w.r.t time't'

$$\frac{\partial \Psi}{\partial t} = -i\, \Psi_0 e^{-i\omega t}$$

$$\frac{\partial \Psi}{\partial t} = (-i\, w\, \Psi) \text{--------------}(3)$$

w.k.t

$$\omega = 2\pi\upsilon \quad ; \text{ but } E = h\,\upsilon\,; \quad \upsilon = \frac{E}{h}$$

$$\omega = 2\pi\frac{E}{h} \text{ --------------}(4)$$

Substitute (4) in (3)

$$\frac{\partial \Psi}{\partial t} = \frac{-i2\,\pi E\Psi}{h} = \frac{-ii2\,\pi E\Psi}{ih} \text{ (multiply \& divide by i)}$$

$$\frac{\partial \Psi}{\partial t} = \frac{-2\,\pi E\Psi}{ih} = \frac{E\Psi}{i\hbar}$$

$$\frac{\partial \Psi}{\partial t}\, i\hbar = E\Psi$$

$$E\Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

Substitute EΨ in time independent wave equation

$$\nabla^2 \psi + \frac{2m(E-V)}{\hbar^2}\,\Psi = 0$$

$$\nabla^2 \psi + \frac{2m(E\Psi - V\Psi)}{\hbar^2} = 0$$

$$\nabla^2 \psi = \frac{-2m(E\Psi - V\Psi)}{\hbar^2}$$

$$\frac{-\hbar^2}{2m}\nabla^2 = E\Psi - V\Psi$$

$$\frac{-\hbar^2}{2m}\nabla^2 + V\Psi = E\Psi \text{----------------}(6)$$

Substitute (5) in (6)

$$\frac{-\hbar^2}{2m}\nabla^2 + V\Psi = i\hbar\frac{\partial \Psi}{\partial t}\text{-----------}(7)$$

$$\left[\frac{-\hbar^2}{2m}\nabla^2 + V\right]\Psi = i\hbar\frac{\partial \Psi}{\partial t}\text{-----------}(8)$$

(8) is the Schrodinger Time dependent wave equation

Here

Hamiltonion operator H= $\left[\frac{-\hbar^2}{2m}\nabla^2 \quad + V\right]$

Energy operator E = $i\hbar\frac{\partial\Psi}{\partial t}$

(8) gives $H\Psi = E\Psi$