2.4 Derivatives of Implicit Functions:

If x and y are connected by a relation $f(x, y) = c$, then it may not be possible to express y as a single valued function of x explicitly. Such a function are called as implicit function. However such functions can be differentiated both side of the equation with respect to x and then solve the resulting equation for $\frac{dy}{dx}$.

Example:

Find $\frac{dy}{dx}$ for the following functions:

a) x^2 + xy – y^2 = 4 b) $x^3 + y^3$ + 3axy = a c) $x^2y^2 + x\sin y = 4$

Solution:

a) $x^2 + xy - y^2 = 4$

Differentiating with respect to x

$$2x + x\frac{dy}{dx} + y - 2y\frac{dy}{dx} = 0$$

$$(x - 2y)\frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx} = \frac{-(2x+y)}{x-2y}$$

b) $x^3 + y^3$ + 3axy = a

Differentiating with respect to x

$$3x^2 + 3y^2\frac{dy}{dx} + 3ax\frac{dy}{dx} + 3ay = 0$$

$$(3y^2 + 3ax)\frac{dy}{dx} = -3x^2 - 3ay$$

$$\frac{dy}{dx} = \frac{-3x^2-3ay}{3y^2+3ax}$$

c) $x^2y^2 + x \sin y = 4$

Differentiating with respect to x

$$x^2.2y\frac{dy}{dx} + 2xy^2 + x.\cos y\frac{dy}{dx} + \sin y = 0$$

$$(2x^2y + x\cos y)\frac{dy}{dx} = -2xy^2 - \sin y$$

$$\frac{dy}{dx} = -\left[\frac{2xy^2+\sin y}{2x^2y+x\cos y}\right]$$

Example :

If $xy = c^2$ then show that $x^2\frac{d^2y}{dx^2} + \frac{xdy}{dx} - y = 0$

Solution:

Given, $xy = c^2$

Differentiating with respect to x,

$$x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

Derivatives of Logarithmic Functions:

$$\frac{d}{dx}(log_a x) = \frac{1}{x\, log_e a}$$

$$\frac{d}{dx}(log_e x) = \frac{1}{x}$$

$$\frac{d}{dx}(log_e u) = \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}log_e|x| = \frac{1}{x}$$

$$\frac{d}{dx}(a^b) = 0$$

$$\frac{d}{dx}[a^{g(x)}] = a^{g(x)}[log_e a]\, g'(x)$$

Example :

Find $\frac{dy}{dx}$ if, (a) y = x^x (b) y = $x^{cos\, x}$ (c) $x^y = y^x$ (d) y = $x^{x^{x^{\cdot^{\infty}}}}$ (e) $x^y = e^{x-y}$

Solution:

(a) $y = x^x$

Taking logarithms on both sides

Log y = log x^x

Log $y = x\, log\, x$

Differentiating with respect to x

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + log\, x$$

$$\Rightarrow \frac{dy}{dx} = y\,(1 + log\, x)$$

(b) $y = x^{\cos x}$

Taking logarithms on both sides

log y = log $x^{\cos x}$

log y = $\cos x \log x$

Differentiating with respect to x

$\Rightarrow \frac{1}{y}\frac{dy}{dx}$ = cos x $\frac{1}{x}$ + log x (- sin x)

$\Rightarrow \frac{dy}{dx}$ = y [$\frac{\cos x}{x}$ – sin x log x]

$\Rightarrow \frac{dy}{dx} = x^{\cos x}[\frac{\cos x - x \sin x \log x}{x}]$

(c) $x^y = y^x$

Taking logarithms on both sides

Differentiating with respect to x

$$y \log x = x \log y$$

$\Rightarrow$y $\frac{1}{x}$ + log x $\frac{dy}{dx}$ = x $\frac{1}{y}\frac{dy}{dx}$ + log y

$\Rightarrow \frac{dy}{dx}$ (log x - $\frac{x}{y}$) = log y - $\frac{y}{x}$

$\Rightarrow \frac{dy}{dx} (\frac{y \log x - x}{y}) = \frac{x \log y - y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y\,(x \log y - y)}{x\,(y \log x - x)}$

(d) $y = x^{x^{x^{\cdot^{\cdot^{\infty}}}}}$

$\Rightarrow$y = x^y

Taking log on both sides

$$Log\, y = y \log x$$

Differentiating with respect to x

$\Rightarrow \frac{1}{y}\frac{dy}{dx}$ = y $\frac{1}{x}$ + log x $\frac{dy}{dx}$

$\Rightarrow \left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}$

$\Rightarrow \left(\frac{1 - y \log x}{y}\right)\frac{dy}{dx} = \frac{y}{x}$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{x(1-y\,og\,x)}$$

(e) $x^y = e^{x-y}$

Taking log on both sides

$$y\log x = (x - y)\log e$$

$$y\log x = x - y$$

$$y(\log x + 1) = x$$

$$\Rightarrow y = \frac{x}{1+\log x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+\log x)(1)-x\left(\frac{1}{x}\right)}{(1+\log x)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

Exercise

1. Find $\frac{dy}{dx}$ for the following implicit functions

(i) $x^2 + y^2 = a^2$ **Ans:** $\frac{dy}{dx} = \frac{-x}{y}$

(ii) $x^2 + y^2 + 2gx + 2fy + c = 0$ **Ans:** $\frac{dy}{dx} = \frac{-x-g}{y+f}$

(iii) $\cos x + \sin y = a$ **Ans:** $\frac{dy}{dx} = \frac{e^x}{1- xe^y}$

(iv) $y = \text{a} + xe^y$ **Ans:** $\frac{dy}{dx} = \frac{e^x}{1- xe^y}$

(v) $x\sin y + y\sin x = c$ **Ans:** $\frac{dy}{dx} = \frac{-(\sin y+y\cos x)}{(x\cos y+\sin x)}$

2. Find y′′ by implicit differentiation

(i) $x^3 + y^3 = 1$ **Ans:** $y'' = \frac{-2x}{y^5}$

(ii) $xy + e^y = e$ at $x = 0$ **Ans:** $y'' = \frac{1}{e^2}$

(iii) $y^2 = x^2 + 2x$ **Ans:** $y'' = \frac{[\,y^2-(x+1)^2}{y^3}$

3.Find y′ using logarithmic differentiation

(i) $y = 10^{3x-1}$ **Ans:** $y' = 3\,[10^{3x-1}]\,\log_e 10$

(ii) $y = (sin\ x)^x$ **Ans:** $y' = (sin\ x)^x$ [x cot x + log sin x]

(iii) $y = x^{sin\ x}$ **Ans:** $y' = (sin\ x)^x\ [\frac{sin\ x}{x}$ + log x cos x]

(iv) $(sin\ x)^{cos\ y} = (sin\ y)^{cos\ x}$ **Ans:** $y' = \frac{sin\ x\ (log\ sin\ y)+cos\ y\ cot\ x}{sin\ y\ (log\ sin\ x)+cos\ x\ cot\ y}$

(v) $y = (sin\ x)^{(sin\ x)^{(sin\ x)^{\cdot^{\cdot^{\infty}}}}}$ **Ans:** $y' = \frac{y^2\ cot\ x}{1-y\ log\ sin\ x}$

Differentiation of parametric functions:

Sometimes x and y are both expressed in terms of a third variable, usually called a parameter.

If $x = f_1(t)$ and $y = f_2(t)$, then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dt}\frac{dt}{dx}$

Example

Find $\frac{dy}{dx}$ **if (a)** $\boldsymbol{x = a\ cos\ \theta, y = b\ sin\ \theta}$ **(b)** $\boldsymbol{x = a\ cos^3\theta, y = a\ sin^3\theta}$

(c) $\boldsymbol{x = log\ t + sin\ t, y = e^t + cos\ t}$

(d) $\boldsymbol{x = a\ (\theta + sin\ \theta), y = a\ (1 - cos\ \theta)}$

Solution:

(a) $x = a\cos\theta\ , y = b\sin\theta$

$x = a\cos\theta$ $y = b\sin\theta$

$\frac{dx}{d\theta} = -a\ sin\ \theta$ $\frac{dy}{d\theta} = b\ cos\ \theta$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\ cos\ \theta}{-\ a\ sin\ \theta}$$

$$= \frac{-\ b}{a}.\ \cot\theta$$

(b) $x = a\ cos^3\theta, y = a\ sin^3\theta$

$x = a\ cos^3\theta$ $y = a\ sin^3\theta$

$\frac{dx}{d\theta} = 3acos^2\ \theta\ (-\ \sin\theta)$ $\frac{dy}{d\theta} = 3\ a\ sin^2\theta\ \cos\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3acos^2\ \theta\ (-\sin\theta)}{3\ a\ sin^2\theta\ \cos\theta}$$

$$= \frac{-\sin\theta}{\cos\theta}$$

$$= -\tan\theta$$

(c) $x = \log t + \sin t, y = e^t + \cos t$

$x = \log t + \sin t \qquad y = e^t + \cos t$

$\frac{dx}{d\theta} = \frac{1}{t} + \cos t \qquad \frac{dy}{d\theta} = e^t - \sin t$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{1}{t} + \cos t}{e^t - \sin t}$$

(d) $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$

$x = a(\theta + \sin\theta) \qquad y = a(1 - \cos\theta)$

$\frac{dx}{d\theta} = a(1 + \cos\theta) \qquad \frac{dy}{d\theta} = a\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(1 + \cos\theta)}{a\sin\theta}$$

$$\frac{dy}{dx} = \tan\frac{\theta}{2}$$

Differentiation for hyperbolic and inverse hyperbolic functions

Formulae:

$Sin\, hx = \frac{e^x - e^{-x}}{2} \qquad \cos hx = \frac{e^x + e^{-x}}{2}$

Hyperbolic Identities:

$$\sinh(-x) = -\sin hx; \cosh(-x) = \cosh x \ ; \cosh^2 x - \sinh^2 x = 1$$

Derivatives of hyperbolic functions

$\frac{d}{dx}(\sin hx) = \cos hx \qquad \frac{d}{dx}(\cos hx) = \sin hx$

$\frac{d}{dx}(\tan hx) = sech^2 x \qquad \frac{d}{dx}(\cot hx) = -cosec\, h^2 x$

$\frac{d}{dx}(cosec\, hx) = -cosec\, hc \cot hx \qquad \frac{d}{dx}(\sec hx) = -\sec hx \tan hx$

Derivatives of inverse hyperbolic functions

$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1} + x^2} \qquad \frac{d}{dx}(\cos h^{-1} x) = \frac{1}{\sqrt{x^2} - 1}$

$\frac{d}{dx}(\tan h^{-1} x) = \frac{1}{1 - x^2} \qquad \frac{d}{dx}(\cot h^{-1} x) = \frac{1}{1 - x^2}$

$\frac{d}{dx}(cosec^{-1}x) = \frac{1}{|x|\sqrt{x^2}+1}$ $\quad$ $\frac{d}{dx}(sec\ h^{-1}x) = \frac{1}{x\sqrt{1}-x^2}$

Example

Find the derivatives for the following functions

a) f(x) = x sin hx – cos hx

b) f(x) = sin h (sox hx)

c) f(x) = $e^{xos\ h\ 3x}$

Solution:

a) $f(x) = x\ sin\ hx - cos\ hx$

$f'(x) = x\ coshx + sin\ hx - sin\ hx$

$f'(x) = x\ coshx$

b) $f(x) = sin\ h\ (cos\ hx)$

$f'(x) = cos\ h\ (coshx)\ x\ sin\ hx$

$f'(x) = sin\ hxcosh(coshx)$

c) $f(x) = e^{xos\ h\ 3x}$

$f'(x) = e^{xos\ h\ 3x}\ (3\ sin\ h3x)$

$f'(x) = 3\ sin\ 3x\ e^{xos\ h\ 3x}$

Example

Find the derivatives of the following inverse functions

a) f(x) = $tan\ h^{-1}\sqrt{x}$

b) f(x) = tan^{-1} (tan hx)

c) f(x) = $tan\ h^{-1}(sin\ x)$

Solution:

a) $f(x) = tan\ h^{-1}\sqrt{x}$

$f'(x) = \frac{1}{1-(\sqrt{x})^2}\frac{1}{2\sqrt{x}}$

$f'(x) = \frac{1}{2\sqrt{x}(1-x)}$

b) $f(x) = tan^{-1}\ (tan\ hx)$

$f'(x) = \frac{1}{1+ tan\,h^2 x} sec\,h^2\,x$

$f'(x) = \frac{1- tan\,h^2 x}{1+ tan\,h^2 x}$

c) $f(x) \;=\; tan\,h^{-1}(sin\,x)$

$f'(x) = \frac{1}{1-(sin\,x)^2}\,cos\,x = \frac{cos\,x}{1- sin^2 x}$

$f'(x) = \frac{cos\,x}{cos^2 x}$

$f'(x) = sec\,x$

Exercise

1. Find the derivatives for the following parametric functions:

(i) $x = at^2\,,\; y = 2at$ **Ans:** $\frac{dy}{dx} = \frac{1}{t}$

(ii) $x = ct\,,\; y = \frac{c}{t}$ **Ans:** $\frac{dy}{dx} = \frac{1}{t^2}$

(ii) $x = a(cos\theta + \theta sin\theta) y = a(sin\theta - \theta cos\theta)$ **Ans:** $\frac{dy}{dx} = tan\theta$

(iv) $x = asec\theta\,, y = btan\theta$ **Ans:** $\frac{dy}{dx} = \frac{b}{a} cosec\theta$

(v) $x = acos^2\theta\,, y = bsin^2\theta$ **Ans:** $\frac{dy}{dx} = -\frac{b}{a}$

2. Find the derivatives for the following hyperbolic functions:

(i) $y = log(coshx)$ **Ans:** $\frac{dy}{dx} = tanhx$

(ii) $y = sech^2(e^t)$ **Ans:** $\frac{dy}{dt} = -2sech^2(e^t)tanh(e^t)e^t$

(iii) $y = \tanh(1 + e^{2x})$ **Ans:** $\frac{dy}{dx} = 2e^{2x}sech^2(1 + e^{2x})$

3. Find the derivatives for the following inverse hyperbolic functions:

(i) $y = cosh^{-1}\sqrt{x}$ **Ans:** $\frac{dy}{dx} = \frac{1}{2\sqrt{x(x-1)}}$

(ii) $y = tanh^{-1}\left(tan\frac{x}{2}\right)$ **Ans:** $\frac{dy}{dx} = \frac{sec\,x}{2}$

(iii) $y = sech^{-1}\sqrt{1 - x^2}$ **Ans:** $\frac{dy}{dx} = \frac{1}{1-x^2}$