

2.4 Transformation of Random Variables:

Let (X, Y) be a continuous two dimensional random variables with JPDP

$f_{XY}(x, y)$. Transform X and Y to new random variables $U = h(x, y), V = g(x, y)$.

Then the joint PDF of (U, V) is given by

$$f_{UV}(u, v) = |J| f_{XY}(x, y)$$

$$\text{where } J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Procedure to find the Marginal pdf of U & V

(1) Take u as the random variable to which the PDF to be computed and take $v = y$. (if not given)

(2) Express x and y in terms of u and v .

$$(3) \text{ Find } J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

(4) Write the JPDP of (U, V) , $f_{UV}(u, v) = |J| f_{XY}(x, y)$

(5) Substitute the values of J, x and y .

(6) Find the range of u and v using the range of x and y .

(7) The PDF of U is $f_U(u) = \int_{v=-\infty}^{v=\infty} f_{uv}(u, v) dv$

(8) The PDF of V is $f_V(v) = \int_{u=-\infty}^{u=\infty} f_{uv}(u, v) du$

Problem based on Transformation of Random Variables

1. If the JPDP $f(x, y)$ is given by $f_{XY}(x, y) = x + y; 0 \leq x, y \leq 1$, find PDF of $U = XY$.

Solution:

Given (X, Y) is a continuous 2D RV defined in $0 < x < 1$ and $0 < y < 1$.

Also Given $f_{xy}(x, y) = x + y; 0 \leq x, y \leq 1$

we have to find the PDF of $u = xy \dots \dots \dots (1)$

let $v = y \Rightarrow y = v$.

$$(1) \Rightarrow u = xv \Rightarrow x = \frac{u}{v}$$

$$\therefore x = \frac{u}{v} \quad y = v$$

$$\frac{\partial z}{\partial u} = \frac{1}{v}; \frac{\partial x}{\partial v} = \frac{-u}{v^2}; \frac{\partial y}{\partial u} = 0; \frac{\partial y}{\partial v} = 1$$

$$J = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

$$J = \frac{1}{v}$$

The JPDF of (U, V) $f_{uv}(u, v) = |J|f_{xy}(x, y)$

$$= \left| \frac{1}{v} \right| (x + y) = \frac{1}{v} \left(\frac{u}{v} + v \right)$$

$$= \frac{u}{v^2} + 1$$

$$f_{uv}(u, v) = \frac{u}{v^2} + 1$$

To find the range for u and v :

We have $0 \leq x \leq 1 \Rightarrow 0 \leq \frac{u}{v} \leq 1$

i.e $0 \leq u \leq v$

Also $0 \leq y \leq 1 \Rightarrow 0 \leq v \leq 1$

On combining the two limits, we get $0 \leq u \leq v \leq 1$

$$\therefore f_{uv}(u, v) = \frac{u}{v^2} + 1, 0 \leq u \leq v \leq 1$$

PDF of U is given by

$$f_U(u) = \int_{v=u}^{v=1} f_{uv}(u, v) dv \quad 0 \leq u \leq v < 1$$

$$= \int_u^1 \left(\frac{u}{v^2} + 1 \right) dv$$

$$= \int_u^1 (uv^{-2} + 1) dv$$

$$= \left[\frac{uv^{-1}}{-1} + v \right]_u^1$$

$$= \left(\frac{u}{-1} + 1 \right) + 1 - u$$

$$= -u + 1 + 1 - u$$

$$= 2 - 2u$$

$$f_U(u) = 2(1 - u) \quad 0 < u < 1$$

2. Let (X, Y) be a continuous two dimensional random. with JPDF $f(x, y) = 4xye^{-(x^2+y^2)} x > 0, y > 0$. Find the PDF of $\sqrt{X^2 + Y^2}$

Solution:

Given (X, Y) is a continuous two dimensional random variables defined in $0 < x < \infty$ and

$$0 < y < \infty$$

Given $f(x, y) = 4xye^{-(x^2+y^2)}, 0 < x < \infty, 0 < y < \infty$

$$\text{let } u = \sqrt{x^2 + y^2} \dots (1)$$

$$\text{Take } v = y \Rightarrow y = v$$

$$(1) \Rightarrow u^2 = x^2 + y^2$$

$$u^2 = x^2 + y^2 \quad y = v$$

$$x^2 = u^2 - v^2 \Rightarrow x = \sqrt{u^2 - v^2}$$

$$x = \sqrt{u^2 - v^2}, y = v$$

$$\frac{\partial x}{\partial u} = \frac{1}{2} \frac{1}{\sqrt{u^2 - v^2}} (2u) = \frac{u}{\sqrt{u^2 - v^2}}; \frac{\partial y}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = \frac{1}{2} \frac{1}{\sqrt{u^2 - v^2}} (-2v) = \frac{-v}{\sqrt{u^2 - v^2}}; \frac{\partial y}{\partial v} = 1 = 1$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{u}{\sqrt{u^2 - v^2}} & \frac{-v}{\sqrt{u^2 - v^2}} \\ 0 & 1 \end{vmatrix}$$

$$J = \frac{u}{\sqrt{u^2 - v^2}}$$

PDF of (U, V) is $f_{UV}(u, v) = |J| f_{XY}(x, y)$

$$= \frac{u}{\sqrt{u^2 - v^2}} 4xy e^{-(x^2 + y^2)}$$

$$= \frac{u}{\sqrt{u^2 - v^2}} 4\sqrt{u^2 - v^2}(v) e^{-u^2}$$

$$f_{UV}(u, v) = 4uve^{-u^2}$$

To find the range for u and v :

We have $x > 0$

We have $y > 0$

$$\sqrt{u^2 - v^2} > 0$$

$$v > 0$$

$$u^2 - v^2 > 0 \quad \Rightarrow 0 < v < \infty$$

$$u^2 > v^2 \Rightarrow u > v$$

$$\Rightarrow v < u$$

On combining the two limits, we get $0 < v < u < \infty$

$$f_{UV}(u, v) = 4uve^{-u^2}, 0 < v < u < \infty$$

PDF of U is given by

$$\begin{aligned} f_U(u) &= \int_{v=0}^{v=u} f_{uv}(u, v) dv \\ &= \int_0^u 4uve^{-u^2} dv \\ &= 4ue^{-u^2} \int_0^u v dv \\ &= 4ue^{-u^2} \left[\frac{v^2}{2} \right]_0^u \\ &= 2u^3 e^{-u^2} \quad 0 < u < \infty \end{aligned}$$

3. The JPDF to two dimensional random variables X and Y is given by,

$$(x, y) = e^{-(x+y)}, x > 0, y > 0. \text{ Find the PDF of } \frac{X+Y}{2}$$

Solution:

Given (X, Y) is a continuous two dimensional random variable defined in

$$0 < x < \infty \text{ and}$$

$0 < y < \infty$. Also given $f(x, y) = e^{-(x+y)}$; $0 < x < \infty, 0 < y < \infty$

let $u = \frac{x+y}{2} \dots \dots \dots (1)$. Take $v = y \Rightarrow y = v$

$$(1) \Rightarrow u = \frac{1}{2}(x + v)$$

$$2u = x + vx = 2u - v$$

$$\therefore x = 2u - v;$$

$$y = v \frac{\partial x}{\partial u} = 2 \frac{\partial x}{\partial v} = -1; \frac{\partial y}{\partial u} = 0; \frac{\partial y}{\partial v} = 1$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2$$

the PDF of (U, V) is $f_{uv}(u, v) = |J|f_{XY}(x, y)$

$$= 2e^{-(x+y)}$$

$$= 2e^{-(2u-v+v)}$$

$$= 2e^{-2u}$$

To find range for u and v :

We have $x > 0 \Rightarrow 2u - v > 0$

i. e., $2u > v \Rightarrow v < 2u$

Also $y > 0 \Rightarrow v > 0$

$$\therefore v < 2u; v > 0$$

$$0 < v < 2u < \infty$$

On combining the two limits, we get $0 < v < 2u < \infty$

$$\therefore f_{UV}(u, v) = 2e^{-2u}, 0 < v < 2u < \infty$$

The PDF of U is

$$\begin{aligned} f_U(u) &= \int_{v=0}^{v=2u} f_{uv}(u, v) dv \\ &= \int_0^{2u} 2e^{-2u} dv \\ &= 2e^{-2u} \int_0^{2u} dv \\ &= 2e^{-2u} [v]_0^{2u} \\ &= 2e^{-2u} (2u) \\ f_u(u) &= 4ue^{-2u}; u > 0 \end{aligned}$$

UNIT STEP FUNCTION:

$$u(x) = 1 \text{ for } x > 0$$

$$u(x) = 0 \text{ for } x < 0$$

1. If X and Y are two independent random variables each normally distributed with mean = 0 and variance σ^2 , find the density function of $R = \sqrt{X^2 + Y^2}$ and $\phi = \tan^{-1}\left(\frac{Y}{X}\right)$

Solution:

Given that X follows $N(0, \sigma)$.

$$\therefore f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}x^2}; -\infty < x < \infty$$

Also Y follows $N(0, \sigma)$.

$$\therefore f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}y^2}; -\infty < y < \infty$$

Since X and Y are independent, $f_{XY}(x, y) = f_X(x)f_Y(y)$

$$= \frac{1}{\sigma^2 2\pi} e^{\frac{-1}{2\sigma^2}(x^2+y^2)}; -\infty < x < \infty, -\infty < y < \infty$$

We have $r = \sqrt{x^2 + y^2}; \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\Rightarrow x = r \cos \theta, y = r \sin \theta,$$

$$\Rightarrow J = r$$

JPDF of (R, ϕ) is $f_{R\phi}(r, \theta) = |J| f_{XY}(x, y)$

$$= r \frac{1}{\sigma^2 2\pi} e^{\frac{-1}{2\sigma^2}(x^2+y^2)}$$

$$= \frac{r}{\sigma^2 2\pi} e^{\frac{-1}{2\sigma^2}r^2}$$

To find the range for r and θ :

We have $-\infty < x < \infty, -\infty < y < \infty$ t.e entire XY plane.

The entire XY plane is transformed into $x = r \cos \theta, y = r \sin \theta$

i.e the entire XY plane is transformed into $x^2 + y^2 = r^2$ (a circle of infinite radius)

Whole region is transformed into a circle of infinite radius.

$$\therefore 0 \leq r < \infty, 0 \leq \theta \leq 2\pi$$

$$\therefore f_{R\phi}(r, \theta) = \frac{r}{\sigma^2 2\pi} e^{\frac{-1}{2\sigma^2}r^2} \quad 0 \leq r < \infty, 0 \leq \theta \leq 2\pi$$

The PDF of R is

$$f_R(r) = \int_{r=0}^{\infty} f_{r\theta}(r, \theta) d\theta$$

$$= \int_0^{2\pi} \frac{r}{\sigma^2 2\pi} e^{\frac{-1}{2\sigma^2}r^2} d\theta$$

$$= \frac{r}{\sigma^2 2\pi} e^{\frac{-1}{2\sigma^2}r^2} \int_0^{2\pi} d\theta$$

$$= \frac{r}{\sigma^2 2\pi} e^{\frac{-1}{2\sigma^2} r^2} [\theta]_0^{2\pi}$$

$$f_R(r) = \frac{r}{\sigma^2} e^{\frac{-1}{2\sigma^2} r^2}; 0 \leq r < \infty$$

The PDF of ϕ is

$$f_\phi(\theta) = \int_{r=0}^{\infty} f_{r\theta}(r, \theta) dr$$

$$= \int_0^{\infty} \frac{r}{\sigma^2 2\pi} e^{\frac{-1}{2\sigma^2} r^2} dr$$

$$= \frac{1}{\sigma^2 2\pi} \int_0^{\infty} r e^{\frac{-1}{2\sigma^2} r^2} dr$$

$$\text{Put } \frac{1}{2\sigma^2} r^2 = t$$

$$\frac{1}{2\sigma^2} 2r dr = dt$$

$$r dr = \sigma^2 dt$$

There is no change on the limits

$$f_0(\theta) = \frac{1}{\sigma^2 2\pi} \int_0^{\infty} e^{-t} \sigma^2 dt$$

$$= \frac{1}{2\pi} \left[\frac{e^{-t}}{-1} \right]_0^{\infty}$$

$$= \frac{1}{2\pi} (0 + 1)$$

$$f_{\phi}(\theta) = \frac{1}{2\pi} \quad 0 \leq \theta \leq 2\pi$$

2. The random variables X and Y each follows an exponent distribution with parameter 1 and are independent. Find the PDF of $U = X - 1$

Solution:

Given X and Y follows exponential distribution with parameter with $\lambda = 1$

$$\therefore f_x(x) = \lambda e^{-\lambda x}; x > 0$$

$$= e^{-x}$$

$$f_y(y) = e^{-y}; y > 0$$

Since X and y are independent,

$$f_{XY}(x, y) = f_x(x)f_y(y)$$

$$= e^{-x}e^{-y}$$

$$= e^{-(x+y)}$$

let $u = x - y$ (1) Take $v = y \Rightarrow y = v$

$$(1) \Rightarrow u = x \quad v = y \Rightarrow x = u + v$$

$$x = u + v; y = v$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

The JPDF of (U, V) is $f_{uv}(u, v) = |J| f_{xy}(x, y)$

$$= (1) e^{-(x+y)} = e^{-(u+v+v)}$$

$$= e^{-(u+2v)}$$

To find the range for u and v :

$$\text{We have } x > 0 \Rightarrow u + v > 0 \Rightarrow u > -v$$

$$\text{and } y > 0 \Rightarrow v > 0$$

$$\therefore f_{uv}(u, v) = e^{-(u+2v)} \quad u > -v, v > 0$$

The PDF of U is

$$f_u(u) = \int f(u, v) dv$$

Since there are two slopes, the region is divided into two sub regions R_1 and R_2

In R_1 :

$$\text{At } P_1, v = -u; \text{ At } Q_1, v = 0$$

In R_2 :

$$\text{At } P_2, v = 0; \text{ At } Q_2, v = \infty$$

In R_1 :

$$f_U(u) = \int_{v=-4}^{\infty} f(u, v) dv$$

$$= \int_{-u}^{\infty} e^{-(u+2v)} dv$$

$$= \int_{-u}^{\infty} e^{-u} e^{-2v} dv$$

$$= e^{-u} \int_{-u}^{\infty} e^{-2v} dv$$

$$= e^{-u} \left[\frac{e^{-2v}}{-2} \right]_{-u}^{\infty}$$

$$= e^{-u} \left[0 - \frac{e^{2u}}{-2} \right]$$

$$= \frac{e^u}{2}; u < 0$$

In R_2

$$f_U(u) = \int_{v=0}^{\infty} e^{-u} f(u, v) dv$$

$$= \int_0^{\infty} e^{-(u+2v)} dv$$

$$= \int_0^{\infty} e^{-u} e^{-2v} dv$$

$$= \int_0^{\infty} e^{-u} \left[0 - \frac{1}{-2} \right]$$

$$= \frac{e^{-u}}{2}; u > 0$$

$$= e^{-u} \int_0^{\infty} e^{-2v} dv$$

$$= e^{-u} \left[\frac{e^{-2v}}{-2} \right]_0^{\infty}$$

$$f_U(u) = \begin{cases} \frac{e^u}{2} & u < 0 \\ \frac{e^{-u}}{2} & u > 0 \end{cases}$$

