Geometric Distribution

A Discrete random variable "X" is said to follow Geometric distribution, if it assumes only non – negative value4s and its probability mass function is given by $P(X = x) = p(x) = q^{x-1}p, x = 1, 2, \ldots, 0 < p \leq 1$ where $q = 1 - p$.

Why $P(X = x) = p(x) = q^{x-1}p$ is called Geometric Distribution.

Solution:

Putting x = 1, 2, 3, . . in $P(X = x) = p(x) = q^{x-1}p$

We get $q^0p, qp, q^2p, q^3p, \ldots$ which are the various terms of Geometric progression. Hence it is known as Geometric distribution.

State another form of Geometric distribution.

Solution:

The another form of Geometric distribution is

$$P(X = x) = p(x) = q^x p, x = 0, 1, 2, \ldots 0 < p \leq 1 \text{ where } q = 1 - p$$

Find the MGF of geometric distribution and hence find Mean and variance.

Solution:

Geometric distribution is $p(x) = q^{x-1}p; x = 1,2, \ldots \ldots, \infty$

$$M_X(t) = E[e^{tx}] = \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} pq^{x-1}$$

$$= pq^{-1}\sum_{x=1}^{\infty} e^{tx} q^{x} = pq^{-1}\sum_{x=1}^{\infty} (qe^{t})^{x}$$

$$= pq^{-1}[qe^{t} + (qe^{t})^2 + (qe^{t})^3 + \cdots]$$

$$= pq^{-1}qe^{t}[1 + qe^{t} + (qe^{t})^2 + \cdots]$$

$$= pe^{t}[1 - qe^{t}]^{-1}$$

$$M_X(t) = \frac{pe^t}{1-qe^t}$$

To find the mean value of:

$$\text{Mean } E(X) = \left[\frac{d}{dt}[M_X(t)]\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{pe^t}{1-qe^t}\right)\right]_{t=0}$$

$$= \left[\frac{(1-qe^t)pe^t - pe^t(0-qe^t)}{(1-qe^t)^2}\right]_{t=0}$$

$$= \left[\frac{pe^t - pqe^{2t} + pqe^{2t}}{(1-qe^t)^2}\right]_{t=0}$$

$$= \left[\frac{pe^t}{(1-qe^t)^2}\right]_{t=0} \dots\dots\dots (1)$$

$$= \frac{p}{(1-q)^2} = \frac{p}{p^2}$$

$$E(X) = \frac{1}{p}$$

To find variance

$$E(X^2) = \left[\frac{d^2}{dt^2}[M_X(t)]\right]_{t=0}$$

$$= \left[\frac{d^2}{dt^2}\left(\frac{pe^t}{1-qe^t}\right)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{pe^t}{(1-qe^t)^2}\right)\right]_{t=0} \quad \text{From (1)}$$

$$= \left[\frac{(1-qe^t)^2 pe^t - pe^t 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}\right]_{t=0}$$

$$= (1-qe^t)\left[\frac{(1-qe^t)pe^t + 2pqe^{2t}}{(1-qe^t)^4}\right]_{t=0}$$

$$= \left[\frac{(1-qe^t)(pe^t) + 2pqe^{2t})}{(1-qe^t)^3}\right]_{t=0}$$

$$= \left(\frac{p[(1-q)+2q]}{(1-q)^3}\right)$$

$$= p\left[\frac{p+2q}{p^3}\right]$$

$$= p\left[\frac{p+q+q}{-p^3}\right]$$

$$= \frac{(1+q)}{p^2}$$

Variance $= E(X^2) - [E(X)]^2$

$$= \frac{1+q}{p^2} - \frac{1}{p^2}$$

$$= \frac{1+q-1}{p^2} = \frac{q}{p^2}$$

Variance $= \frac{q}{p^2}$

Problems based on Geometric Distribution

1. Suppose that a trainee soldier shoots a target in an independent fashion. If the probability that the target is shot in any one shot is 0.7 what is the (i) Probability that the target would hit on the 10th attempt?(ii) probability that it taken him less than 4 shots? (iii) Probability that it taken him an even number of shots? (iv) Average number of shots needed to hit the target.

Solution:

Given $p = 0.7$

$q = 1 - p = 1 - 0.7 = 0.3$

The Geometric distribution is $P(X = x) = p(x) = q^{x-1}p; x = 1, 2, 3, \ldots$

(i) P(target would hit on the 10th attempt)$= P[x = 10]$

$$= (0.3)^{10-1}(0.7)$$

$$= 0.0000138$$

(ii) P(target would be hit less than 4 shots) $= P(X < 4)$

$$= P(X = 1) + P(X = 2) + P(X = 3)$$

$$= (0.3)^{1-1}(0.7) + (0.3)^{2-1}(0.7) + (0.3)^{3-1}(0.7)$$

$$= 0.9738$$

(iii) P(he would take an even number of shots)

$$= P(X = 2) + P(X = 4) + P(X = 6) + \ldots$$

$$= (0.3)^{2-1}(0.7) + (0.3)^{4-1}(0.7) + (0.3)^{6-1}(0.7) + \ldots$$

$$= (0.3)^{1}(0.7) + (0.3)^{3}(0.7) + (0.3)^{5}(0.7) + \ldots$$

$$= (0.3)^{1}(0.7)[1 + (0.3)^{2} + (0.3)^{4} + \ldots]$$

$$= (0.3)^{1}(0.7)[1 - (0.3)^{2}]^{-1}$$

$$= 0.21[0.91]^{-1}$$

$$= 0.2307$$

(iv) Average number = $E(X) = \frac{1}{p} = \frac{1}{0.7} = 1.4286$

2. Let one copy of a magazine out of 10 copies bears a special prize following geometric distribution. Determine its mean and variance.

Solution:

Given $p = \frac{1}{10}$ and $q = 1 - \frac{1}{10} = \frac{9}{10}$

Mean of Geometric distribution $= \frac{1}{p} = 10$

Variance $= \frac{q}{p^2} = \frac{9}{10} \times 10^2 = 90$

3. If the probability is0.05 that a certain kind of measuring device will show excessive drift, what is the probability that the sixth of these measuring devices tested will be the first to show excessive drift?

Solution:

Given $p = 0.05$ and $q = 1 - 0.05 = 0.95, x = 6$

The Geometric distribution is $P(X = x) = p(x) = q^{x-1}p; x = 1, 2, 3, \ldots$

$$P(X = 6) = p(6) = (0.95)^{6-1}(0.05)$$

$$= 0.0387$$